Intro to Computer Graphics: Math Review

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Contact Information

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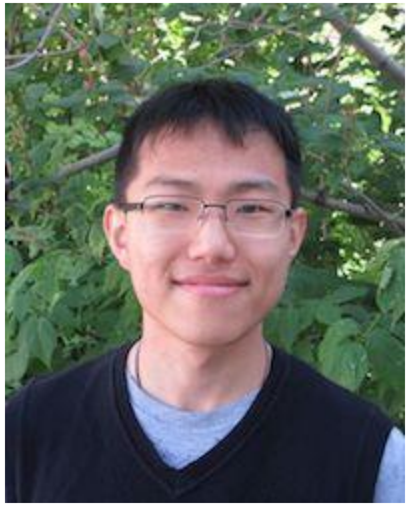
MW 12:00 – 12:50; MW 13:00 – 13:50



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MW 9:00 – 9:50; TR 11:00 – 11:50

Try logging into computer

If you can’t, ask IT after tutorial.

Matrix Operations

Definition of Matrix

$$A_{m\times n} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \\ a_{m0} & a_{m1} & & a_{mn} \end{bmatrix}_{m\times n} = (a_{ij}) \in \mathbb{R}^{m\times n}$$

Matrix: Identity

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}_{n\times n}$$

$$AI = IA = A$$

Matrix: Addition

$$A_{m\times n}+B_{m\times n} = \begin{bmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \\ a_{m0} & & a_{mn} \end{bmatrix}_{m\times n} + \begin{bmatrix} b_{00} & \cdots & b_{0n} \\ \vdots & \ddots & \\ b_{m0} & & b_{mn} \end{bmatrix}_{m\times n}$$

$$= \begin{bmatrix} a_{00}+b_{00} & \cdots & a_{0n}+b_{0n} \\ \vdots & \ddots & \\ a_{m0}+b_{m0} & & a_{mn}+b_{mn} \end{bmatrix}_{m\times n}$$

Matrix: Addition Examples

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

Matrix: Addition Examples

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 4 & 4 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (not\ possible)$$

Matrix: Multiplication with Scalar

$$sA_{m\times n} = s\begin{bmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \\ a_{m0} & & a_{mn} \end{bmatrix}_{m\times n} = \begin{bmatrix} sa_{00} & \cdots & sa_{0n} \\ \vdots & \ddots & \\ sa_{m0} & & sa_{mn} \end{bmatrix}_{m\times n}$$

Matrix: Multiplication with Scalar (Examples)

$$3\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} =$$

$$-2\begin{bmatrix}4\\5\end{bmatrix} =$$

Matrix: Multiplication with Scalar (Examples)

$$3\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}3 & 6\\0 & 3\end{bmatrix}$$

$$-2\begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}-8\\-10\end{bmatrix}$$

Matrix: Multiplication with Matrix (Part 1)

$$A_{m\times n}B_{n\times p} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ \vdots & \vdots & \ddots & \\ a_{m0} & a_{m1} & & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} b_{00} & \cdots & b_{0p} \\ b_{10} & \cdots & b_{1p} \\ \vdots & \ddots & \\ b_{n0} & & b_{np} \end{bmatrix}_{n\times p}$$

$$= \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0p} \\ c_{10} & c_{11} & \cdots & c_{1p} \\ \vdots & \vdots & \ddots & \\ c_{m0} & c_{m1} & & c_{mp} \end{bmatrix}_{m\times p} = C_{m\times p}$$

Matrix: Multiplication with Matrix (Part 2)

$$c_{ij} = \sum_{r=0}^{n} a_{ir} b_{rj}$$

$$\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ \vdots & \vdots & \ddots & \\ a_{m0} & a_{m1} & & a_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} b_{00} & \cdots & b_{0p} \\ b_{10} & \cdots & b_{1p} \\ \vdots & \ddots & \\ b_{n0} & & b_{np} \end{bmatrix}_{n \times p} = \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0p} \\ c_{10} & c_{11} & \cdots & c_{1p} \\ \vdots & \vdots & \ddots & \\ c_{m0} & c_{m1} & & c_{mp} \end{bmatrix}_{m \times p}$$

Matrix: Multiplication with Matrix (Examples)

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} =$$

Matrix: Multiplication with Matrix (Examples)

$$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}$$

$$\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = (not\ possible)$$

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 3\end{bmatrix}$$

Matrix: Determinant (Part 1)

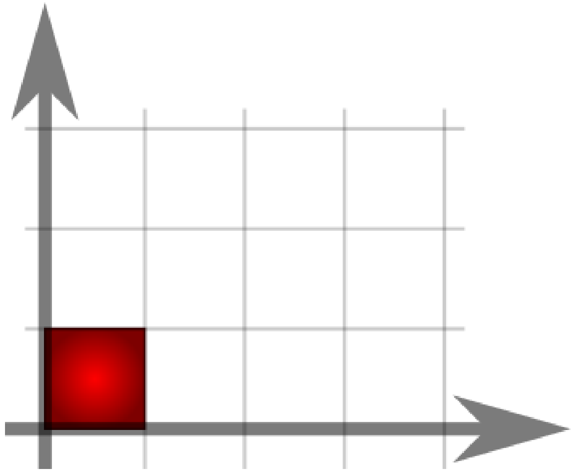
$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$|A| = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Matrix: Determinant (Part 2)

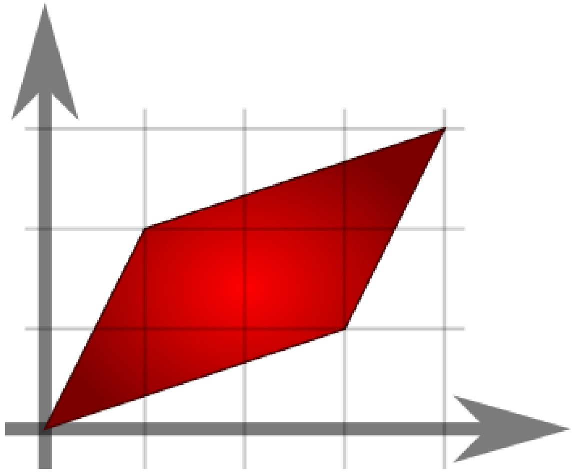
$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$



$$det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$$

Matrix: Determinant (Part 2)

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$



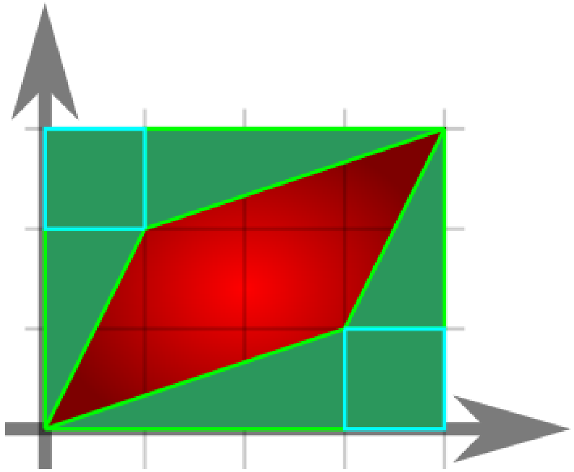
"Volume Scaling Factor"

$$det\left(\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\right) = 5$$

Reference:https://www.youtube.com/watch?v=Ip3X9LOh2dk&vl=en; 3Blue1Brown

Matrix: Determinant (Part 2)

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

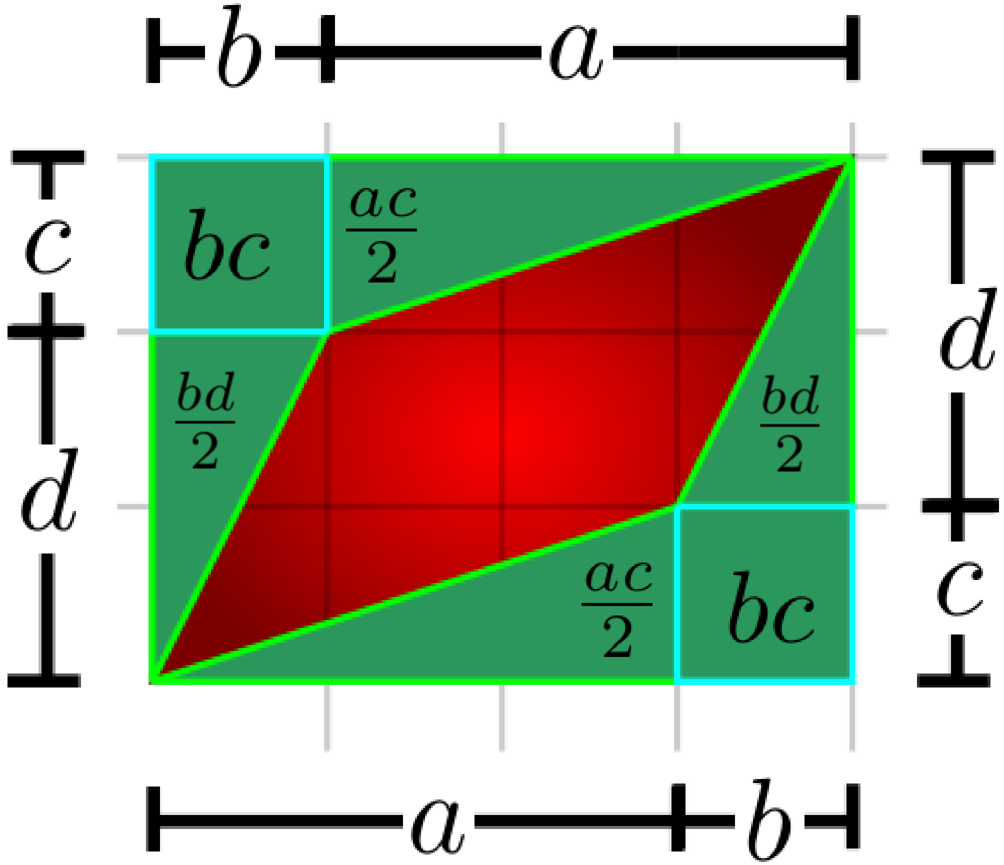


"Volume Scaling Factor"

$$det\left(\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\right) = 5$$

Matrix: Determinant (Part 2)

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

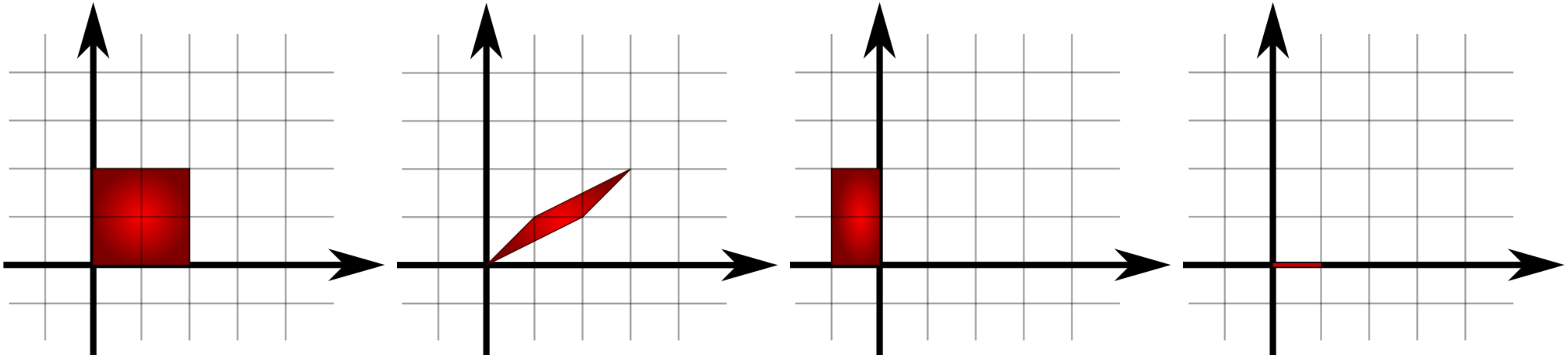


$$det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)(c+d) - 2bc - ac - bd$$

$$= ad - bc$$

Matrix: Determinant (Example)

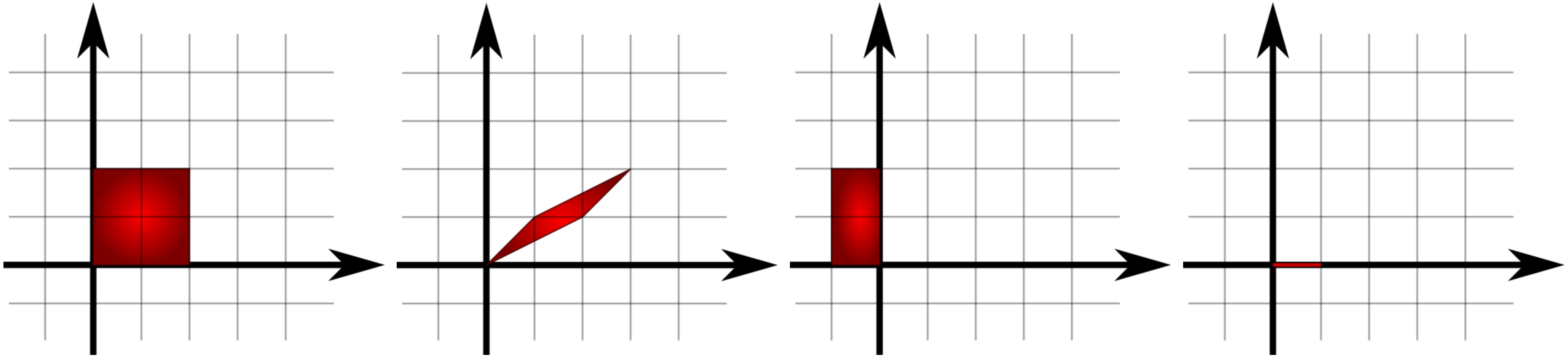
$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$



$$det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = \quad det\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right) = \quad det\left(\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\right) = \quad det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) =$$

Matrix: Determinant (Example)

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$



$$det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 4 \quad det\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right) = 1 \quad det\left(\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\right) = -2 \quad det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$$

Matrix: Inverse

$$AA^{-1} = A^{-1}A = I$$

$$A\vec{x} = \vec{b}$$
$$(A^{-1})A\vec{x} = (A^{-1})\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

Note: Inverse does not exist when determinant of A is zero

Matrix: Transpose

$$A^T = (a_{ij})^T = a_{ji}$$

$$(A_{m\times n})^T = \begin{bmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \\ a_{m0} & & a_{mn} \end{bmatrix}_{m\times n}^T = \begin{bmatrix} a_{00} & \cdots & a_{m0} \\ \vdots & \ddots & \\ a_{0n} & & a_{mn} \end{bmatrix}_{n\times m}$$

Matrix: Transpose (Examples)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^T =$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^T =$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T =$$

Matrix: Transpose (Examples)

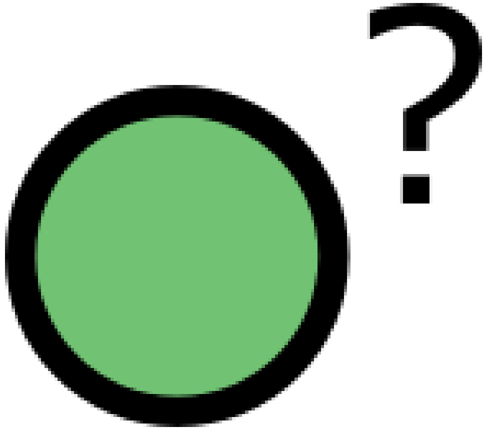
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

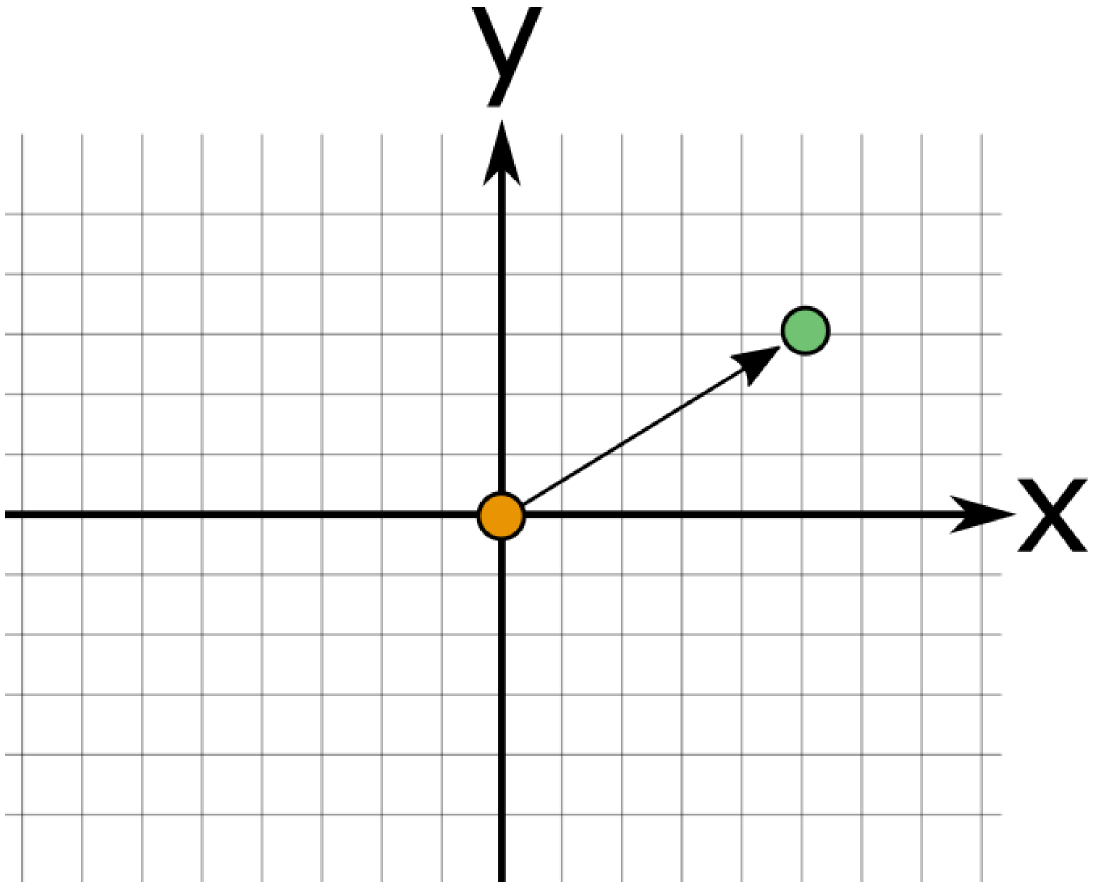
Coordinate System: Cartesian

Basics of Coordinate System



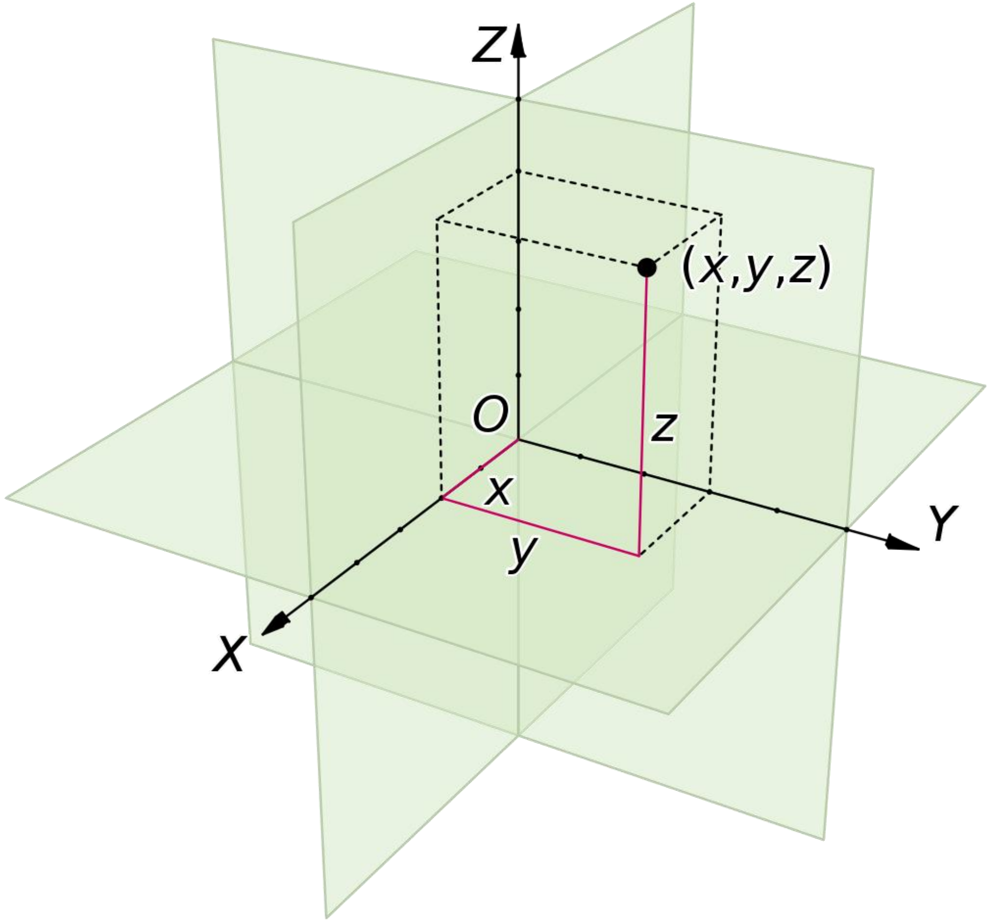
Basics of Coordinate System

Components of Cartesian Coordinate



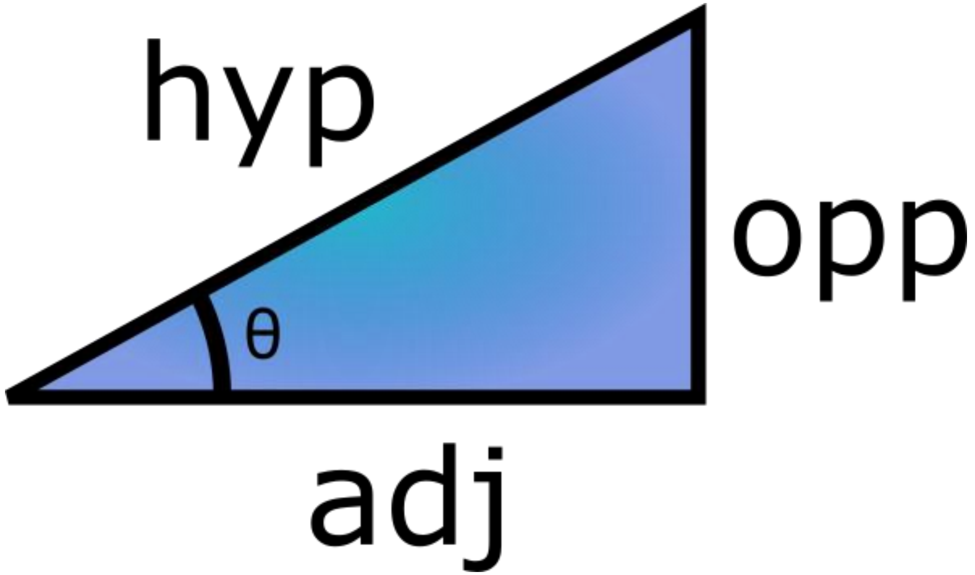
$$\mathbf{v} = \vec{v} = (5, 3) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Visible Dimensions



Trigonometry

Sine and Cosine

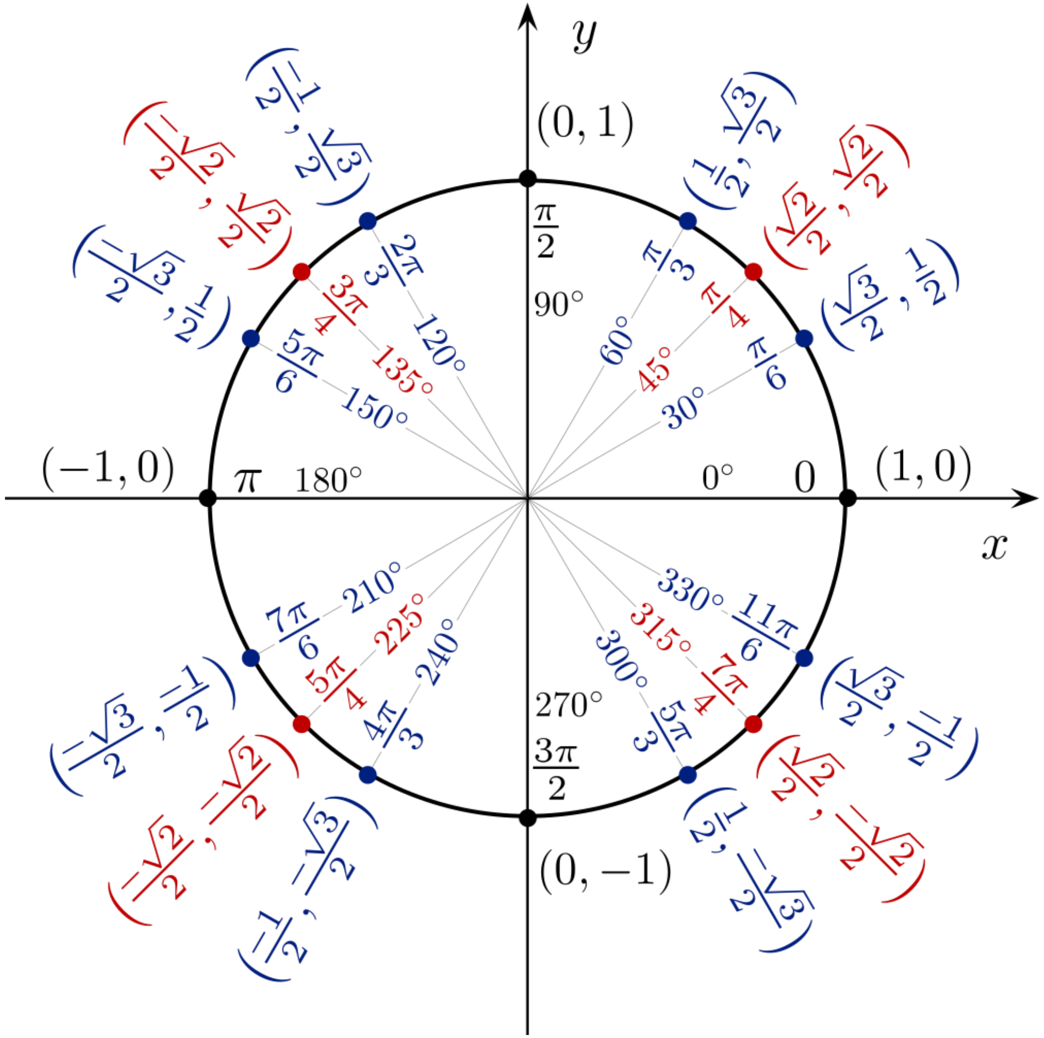


"SOH CAH TOA"

$sin(\theta) = \frac{opp}{hyp}$

$cos(\theta) = \frac{adj}{hyp}$

Unit Circle



Pythagorean theorem:

$$a^2 + b^2 = c^2$$

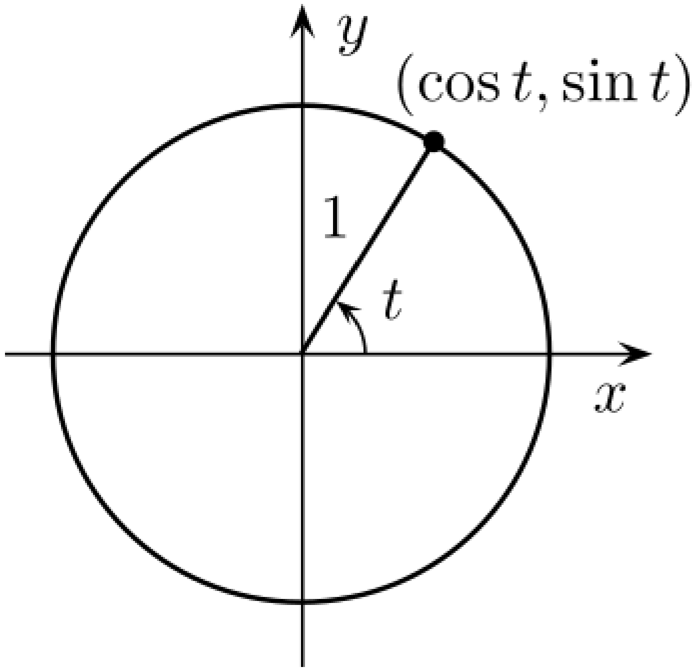
Parametric Equation of Circle:

$$x^2 + y^2 = r^2$$

$$sin(\theta) = \frac{opp}{hyp} = \frac{y}{1} = y$$

$$cos(\theta) = \frac{adj}{hyp} = \frac{x}{1} = x$$

Identities of Trigonometry



Pythagorean identity:

$$cos^2\theta + sin^2\theta = 1$$

Periodicity:

$$cos(\theta + 2\pi) = cos(\theta) \ ; \ sin(\theta + 2\pi) = sin(\theta)$$

Cosine is even & sine is odd:

$$cos(-\theta) = cos(\theta) \ ; \ sin(-\theta) = -sin(\theta)$$

Complementary angles:

$$cos(\frac{\pi}{2} - \theta) = sin(\theta) \ ; \ sin(\frac{\pi}{2} - \theta) = cos(\theta)$$

Additional Property

$$cos(\theta_s + \theta_t) = cos(\theta_s)cos(\theta_t) - sin(\theta_s)sin(\theta_t)$$
$$cos(\theta_s - \theta_t) = cos(\theta_s)cos(\theta_t) + sin(\theta_s)sin(\theta_t)$$
$$sin(\theta_s + \theta_t) = sin(\theta_s)cos(\theta_t) + cos(\theta_s)sin(\theta_t)$$
$$sin(\theta_s - \theta_t) = sin(\theta_s)cos(\theta_t) - cos(\theta_s)sin(\theta_t)$$

Vector Operations

Definition of Vector

$$\vec{v} = magnitude * direction$$

Other Representation of Vector

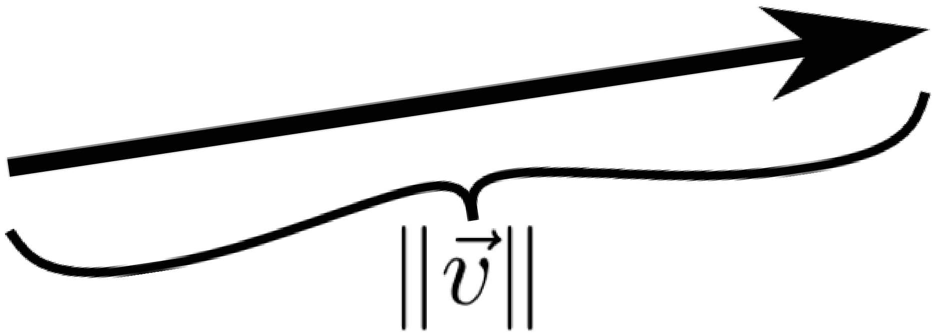
$$\vec{v} = (x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\vec{v} = \vec{o} + t\vec{d}$$

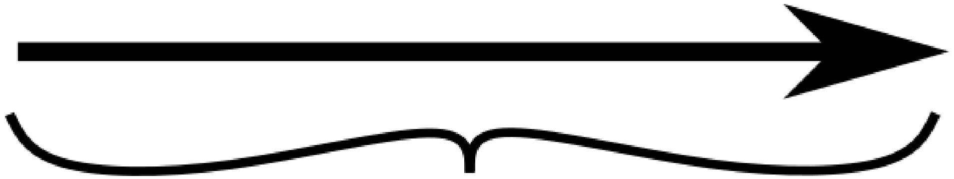
$$\vec{v} = \|\vec{v}\| \cdot \hat{v}$$

Vector: Length

$$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{\vec{v} \cdot \vec{v}}$$



Vector: Normalize

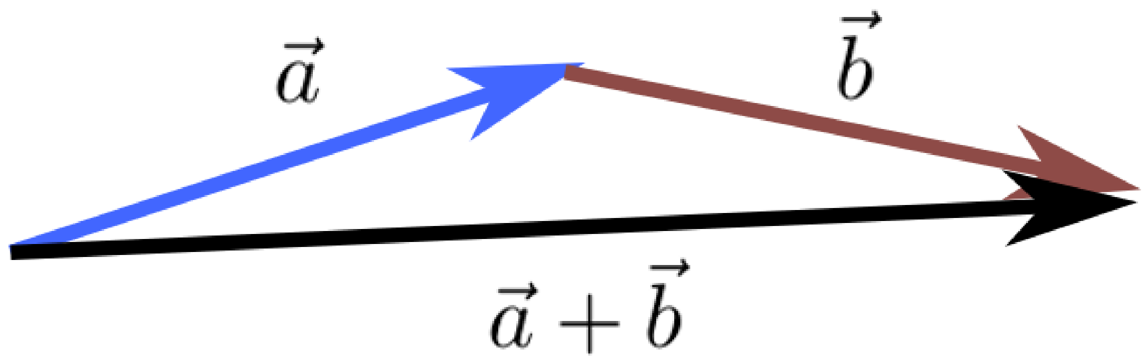


$$\|\vec{v}\| = 1$$

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

Vector: Addition

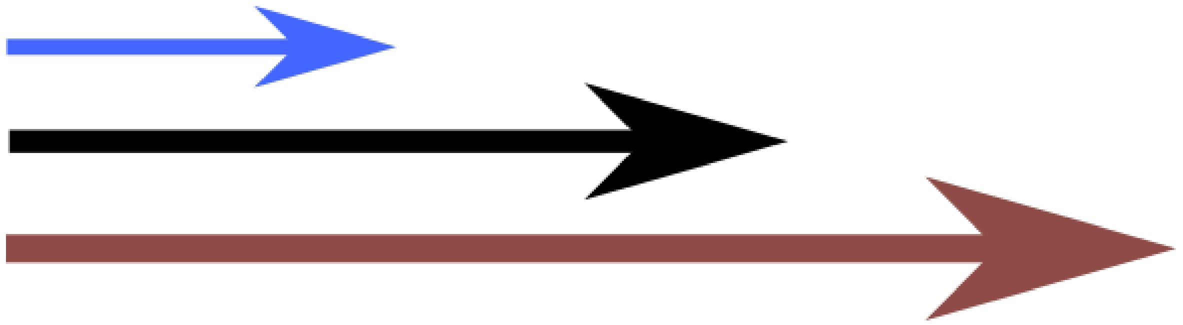
$$\vec{a} + \vec{b} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$



Vector: Scalar Product

$$s\vec{a} = s \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} sa_0 \\ sa_1 \\ sa_2 \end{bmatrix}$$

$$if\ (s < 1):$$
$$if\ (s = 1):$$
$$if\ (s > 1):$$



Vector: Multiplication with Matrix

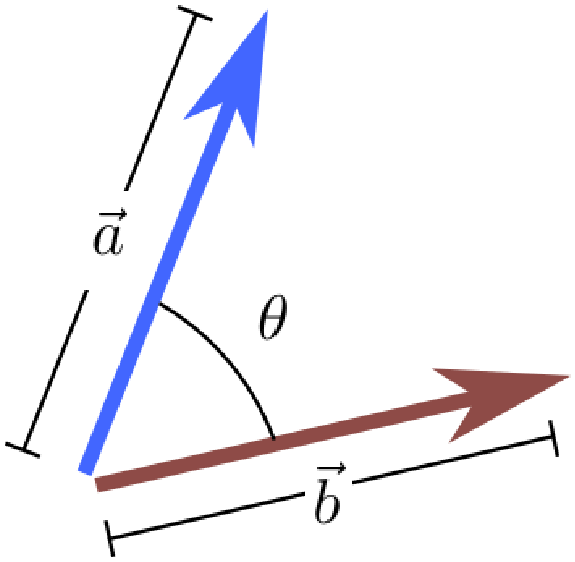
$$A\vec{x} = \vec{b} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

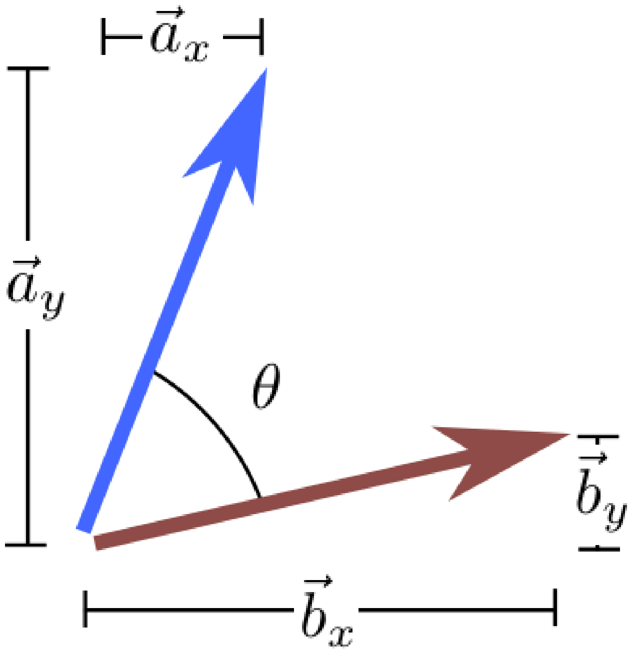
$$= \begin{bmatrix} au + bv + cw \\ du + ev + fw \\ gu + hv + iw \end{bmatrix} \qquad A\vec{x}\ by\ rows$$

$$= u \begin{bmatrix} a \\ d \\ g \end{bmatrix} + v \begin{bmatrix} b \\ e \\ h \end{bmatrix} + w \begin{bmatrix} c \\ f \\ i \end{bmatrix} \qquad A\vec{x}\ by\ columns$$

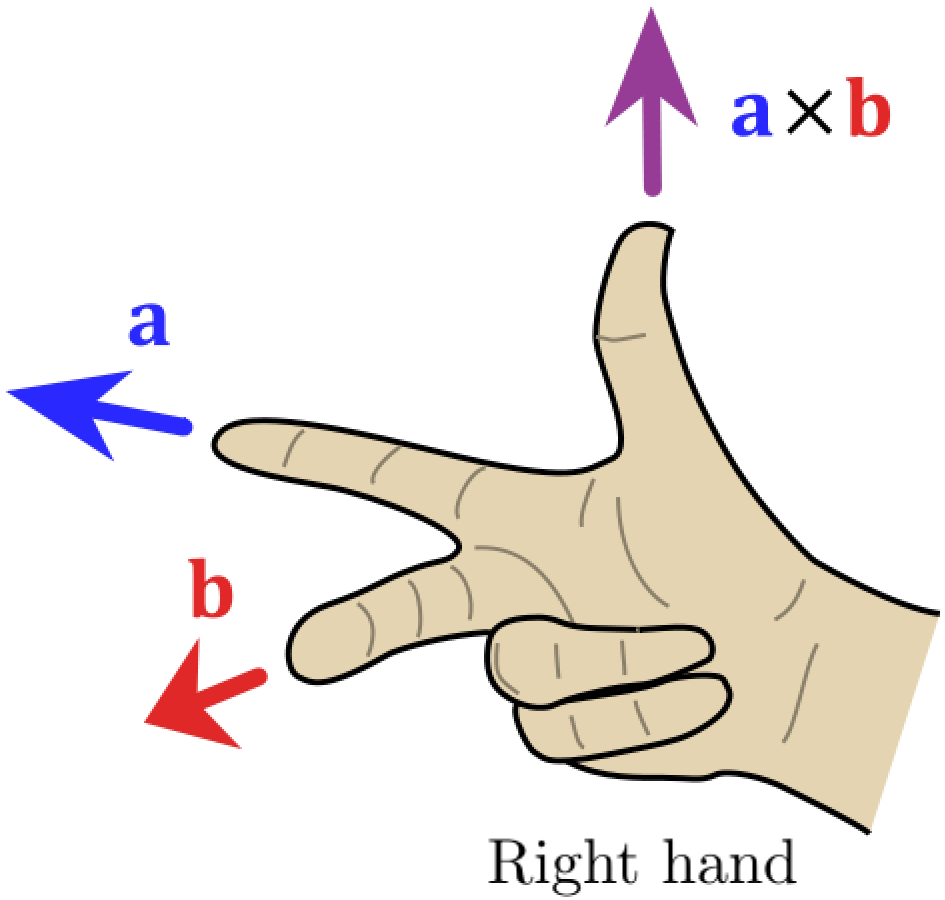
Vector: Dot Product

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta = a_0 b_0 + a_1 b_1 + a_2 b_2$$





Vector: Cross Product (Part 1)

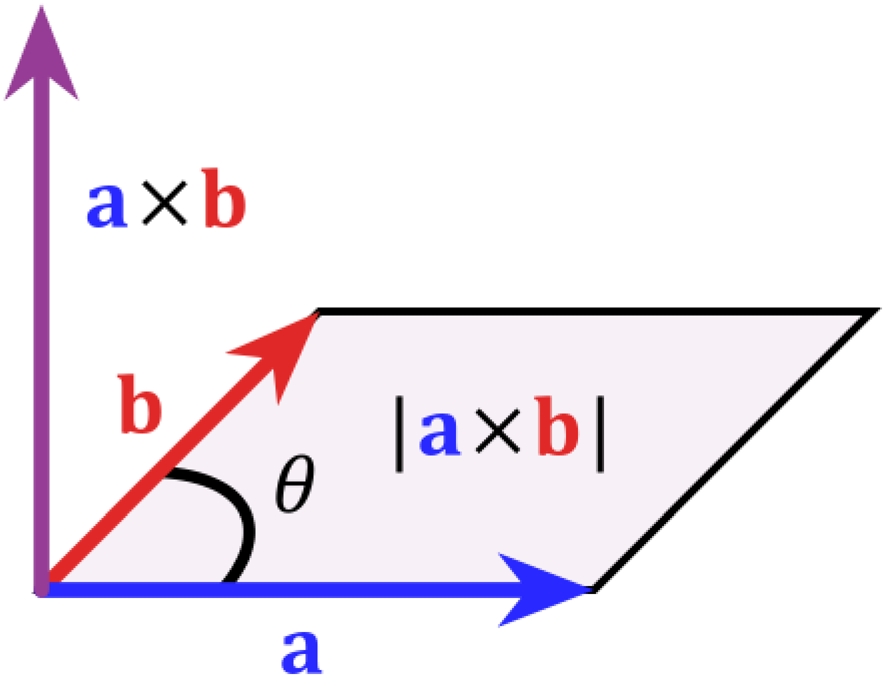


$$\hat{i} = (1,0,0) \qquad \hat{i} \times \hat{j} = \hat{k}$$

$$\hat{j} = (0,1,0) \qquad \hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} = (0,0,1) \qquad \hat{k} \times \hat{i} = \hat{j}$$

Vector: Cross Product (Part 2)



$$\vec{a} \times \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \sin(\theta) \hat{n}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = (a_0\hat{i} + a_1\hat{j} + a_2\hat{k}) \times (b_0\hat{i} + b_1\hat{j} + b_2\hat{k})$$
$$= (a_1 b_2 - a_2 b_1)\hat{i} + (a_2 b_0 - a_0 b_2)\hat{j} + (a_0 b_1 - a_1 b_0)\hat{k}$$

Image Source:https://en.wikipedia.org/wiki/File:Cross_product_parallelogram.svg; License in Public Domain