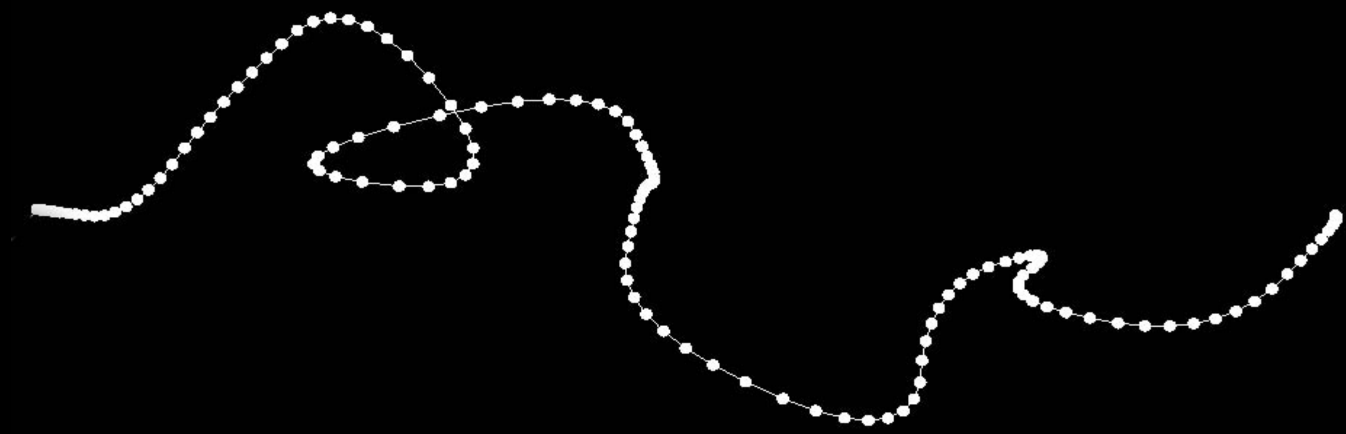
CPSC 453
Subdivision curves



Przemyslaw Prusinkiewicz

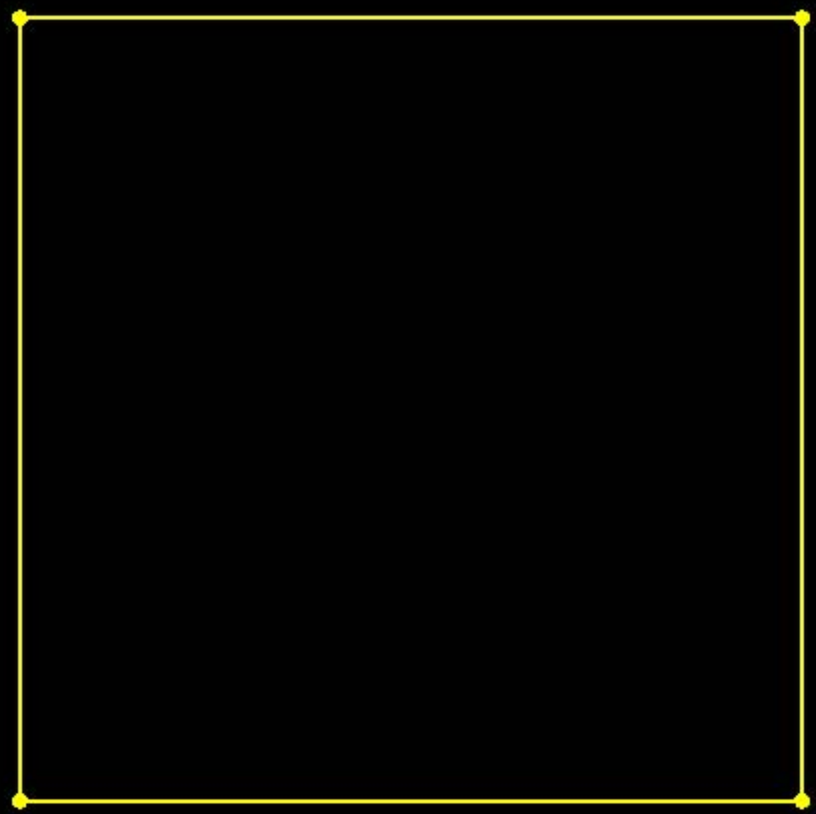
Computer-aided design

Autodesk /Alias 2010

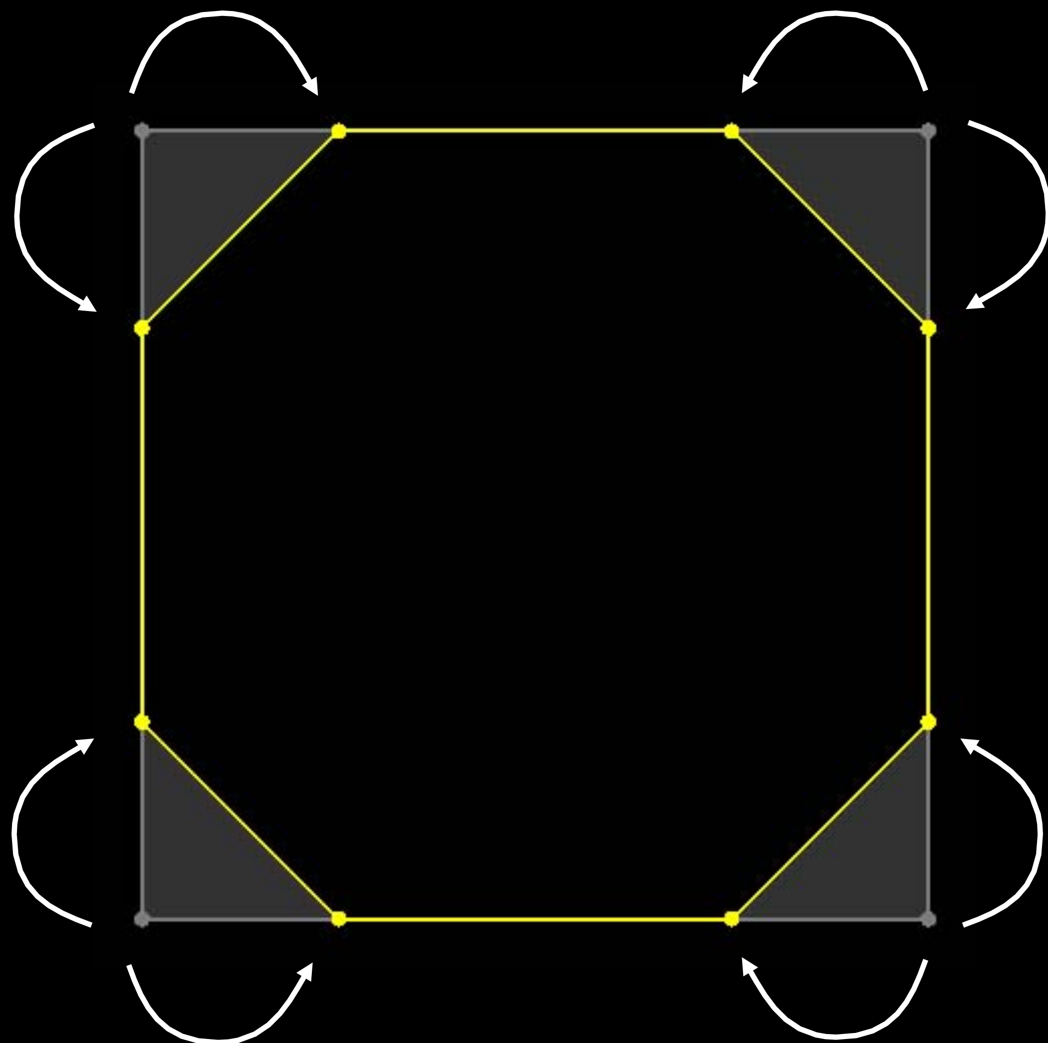
Gentle introduction

Chaikin’s corner-cutting algorithm

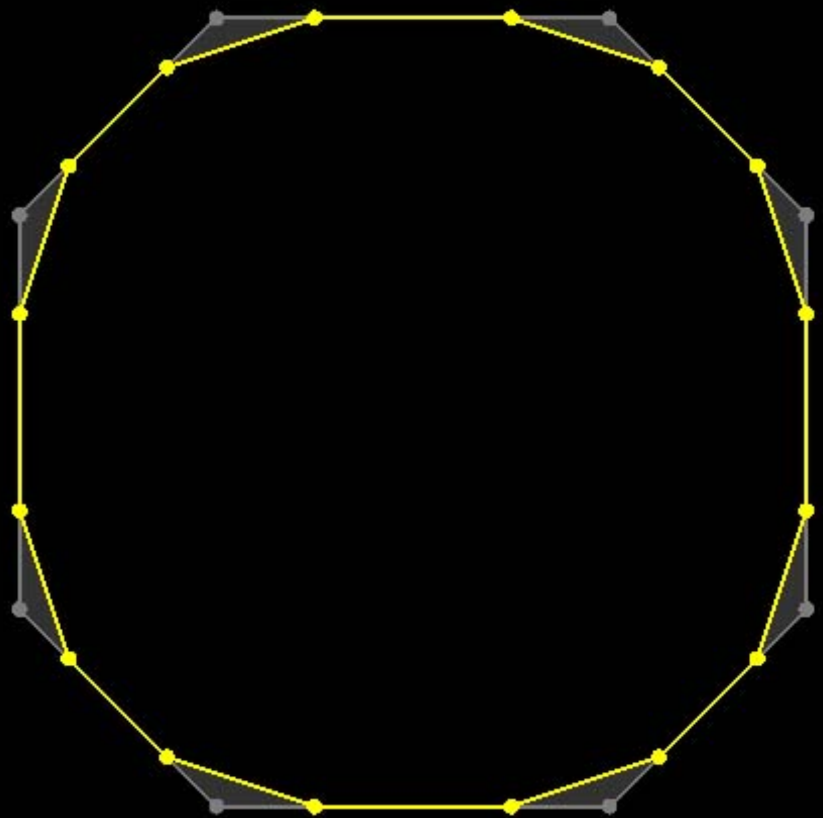
The Chaikin algorithm



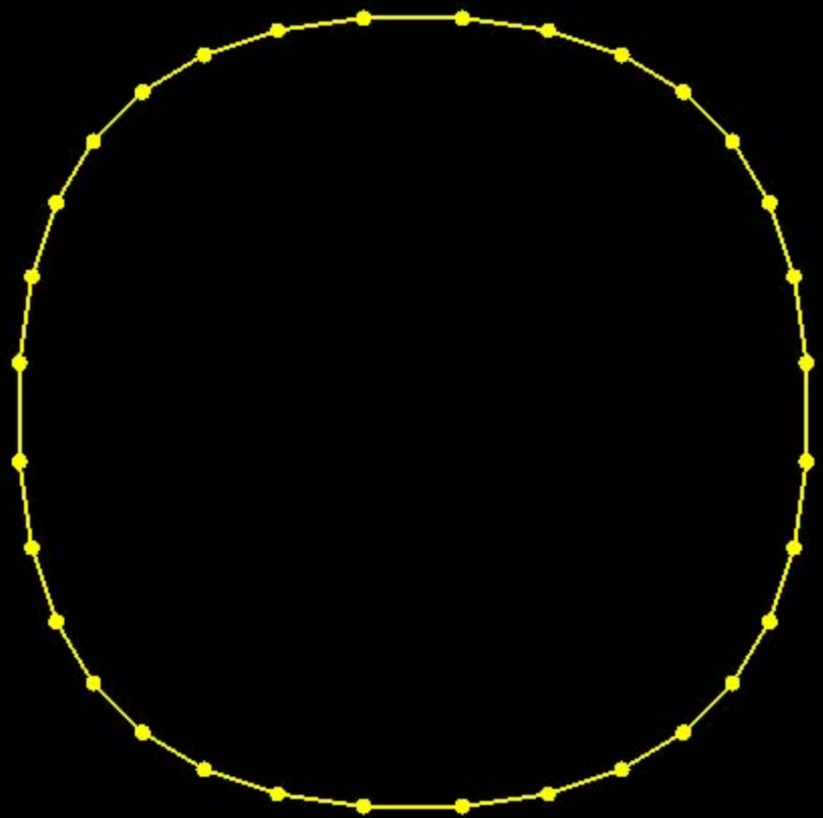
The Chaikin algorithm



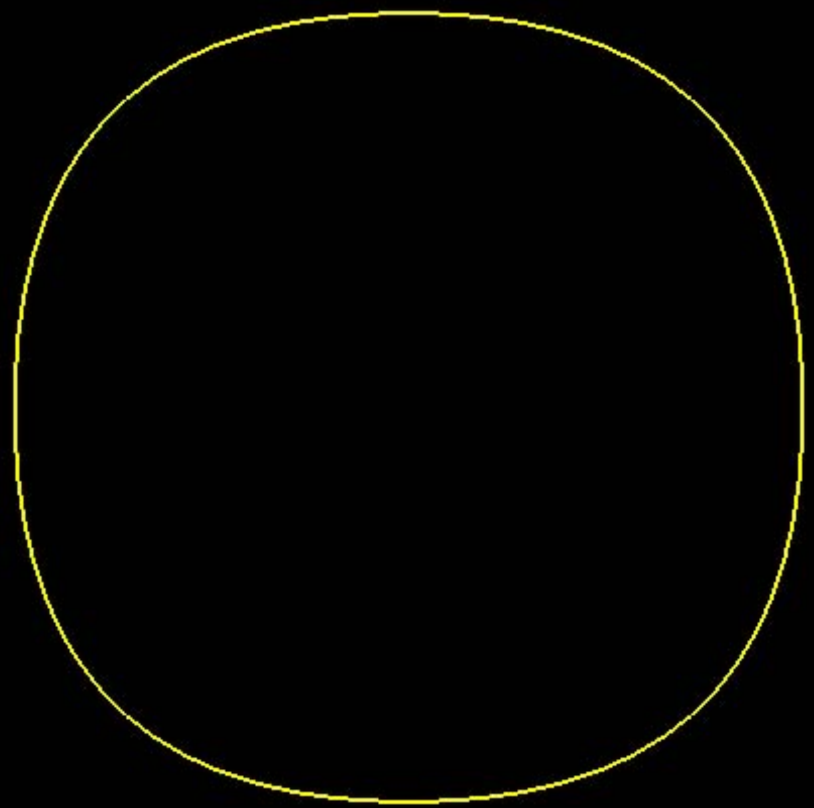
The Chaikin algorithm



The Chaikin algorithm



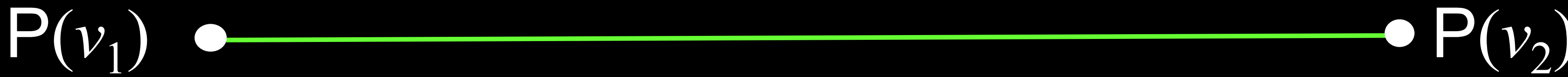
The Chaikin algorithm



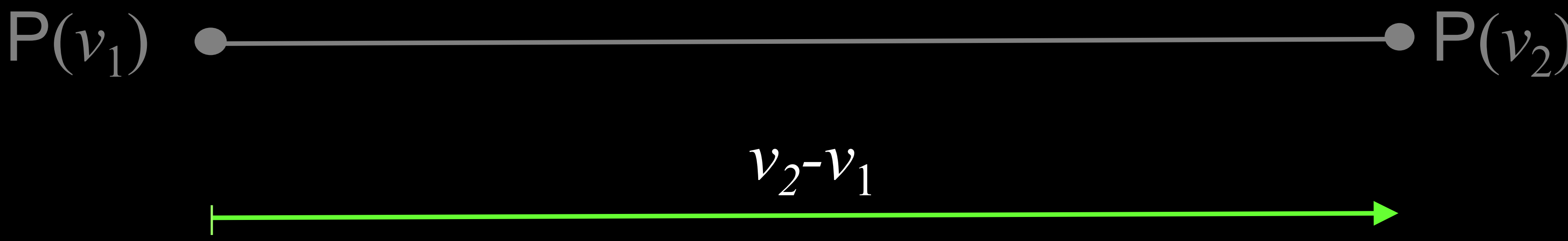
Question

How can we formally describe the Chaikin algorithm?

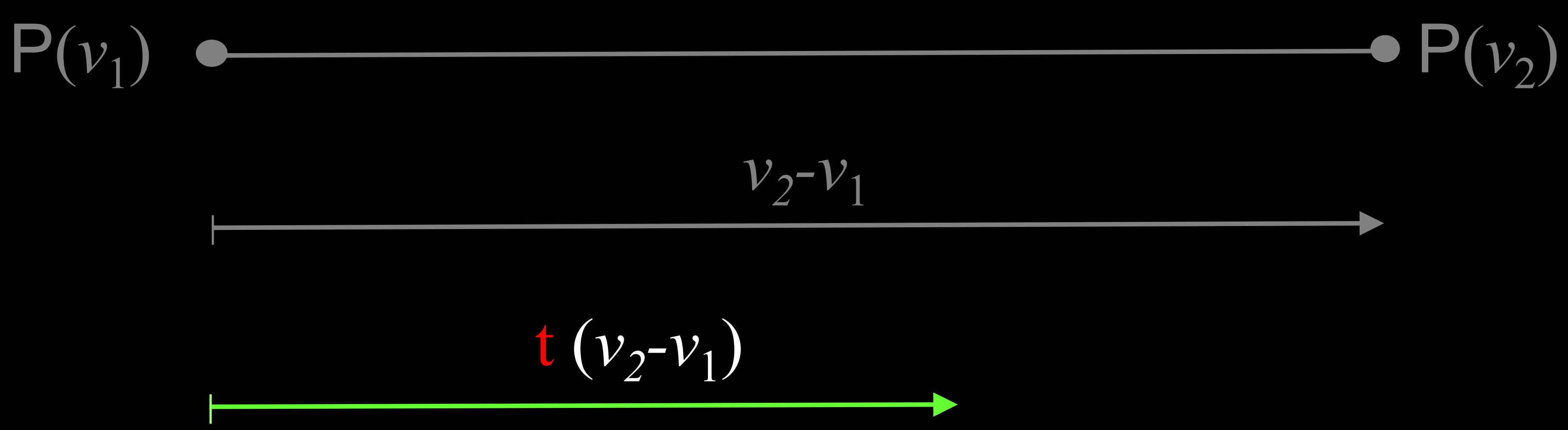
Affine combination of points



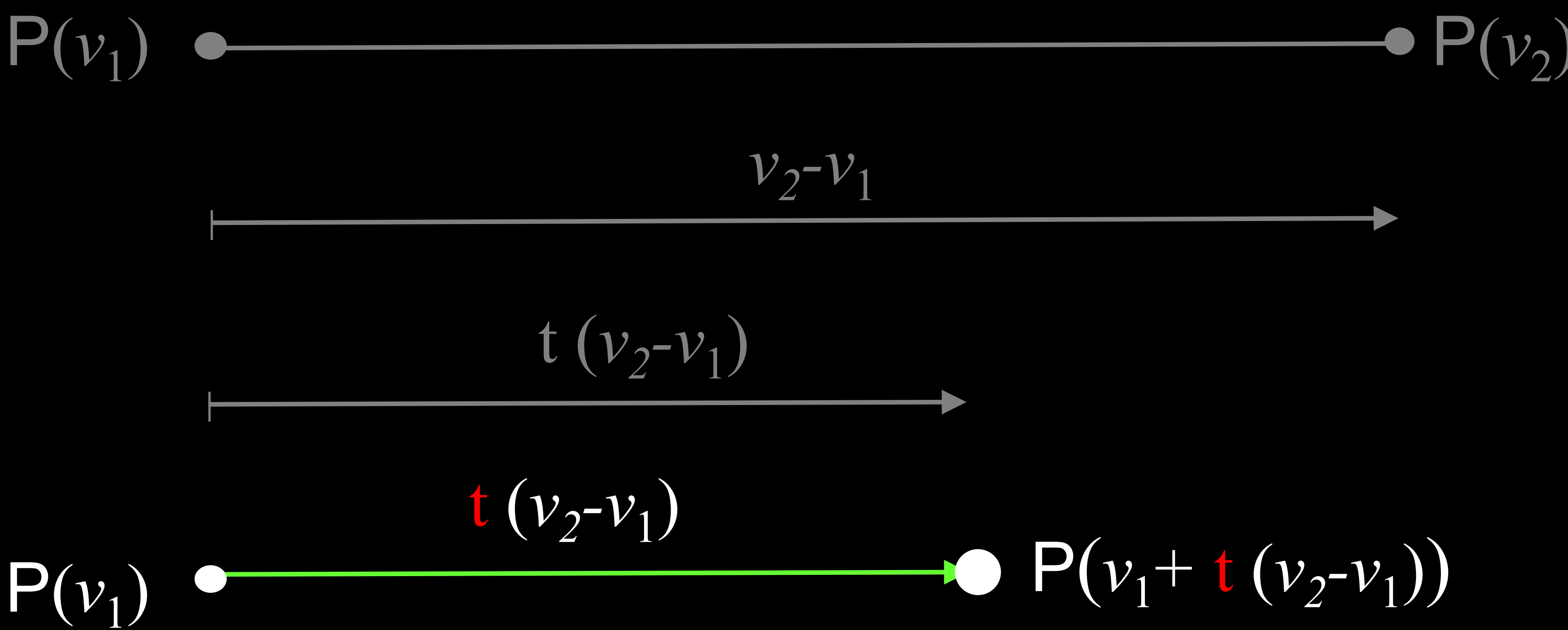
Affine combination of points



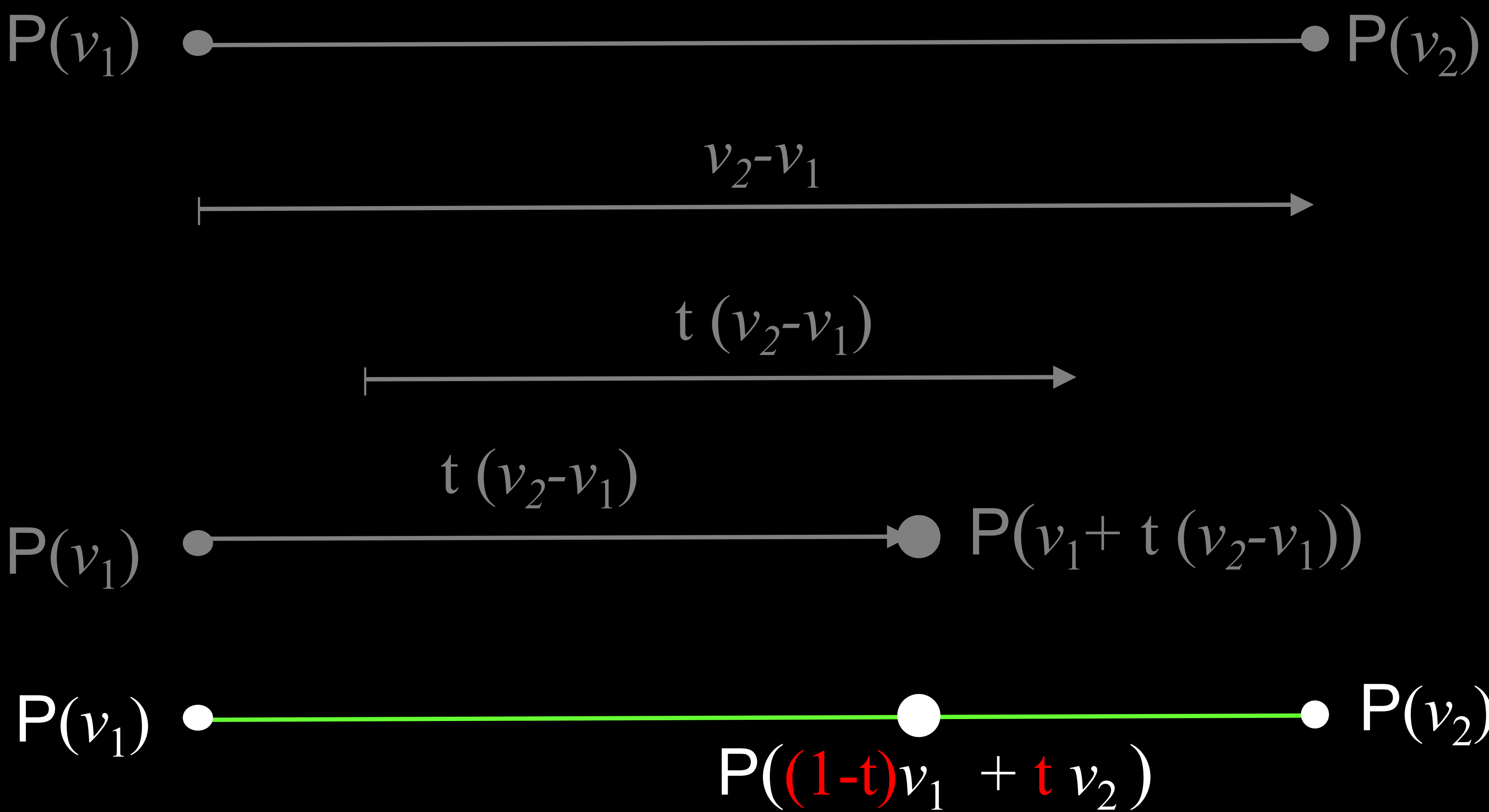
Affine combination of points



Affine combination of points

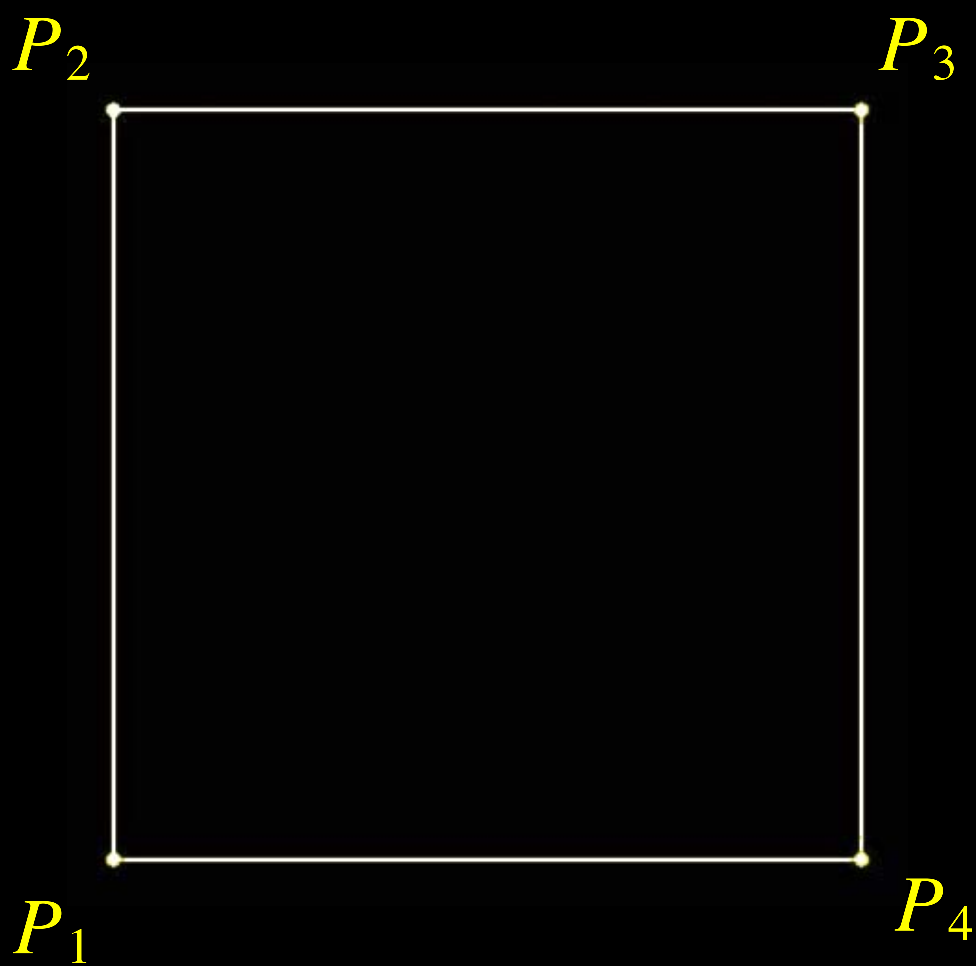


Affine combination of points

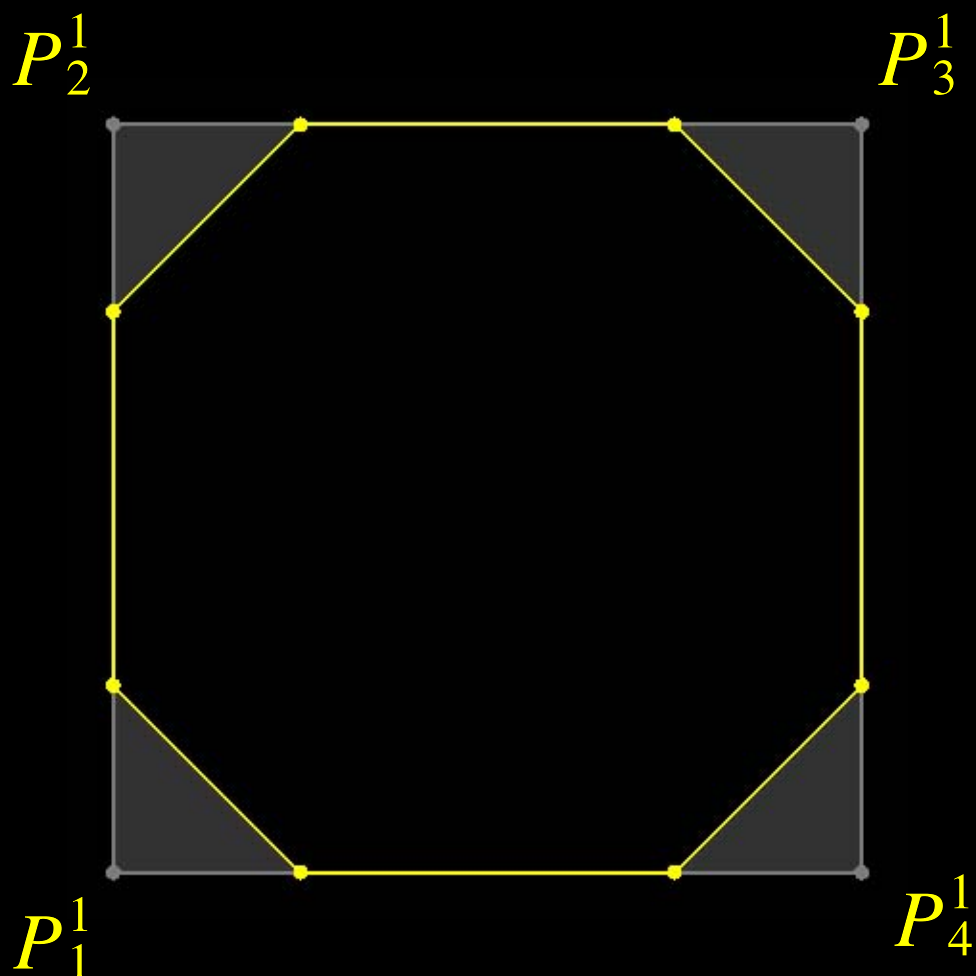


Example

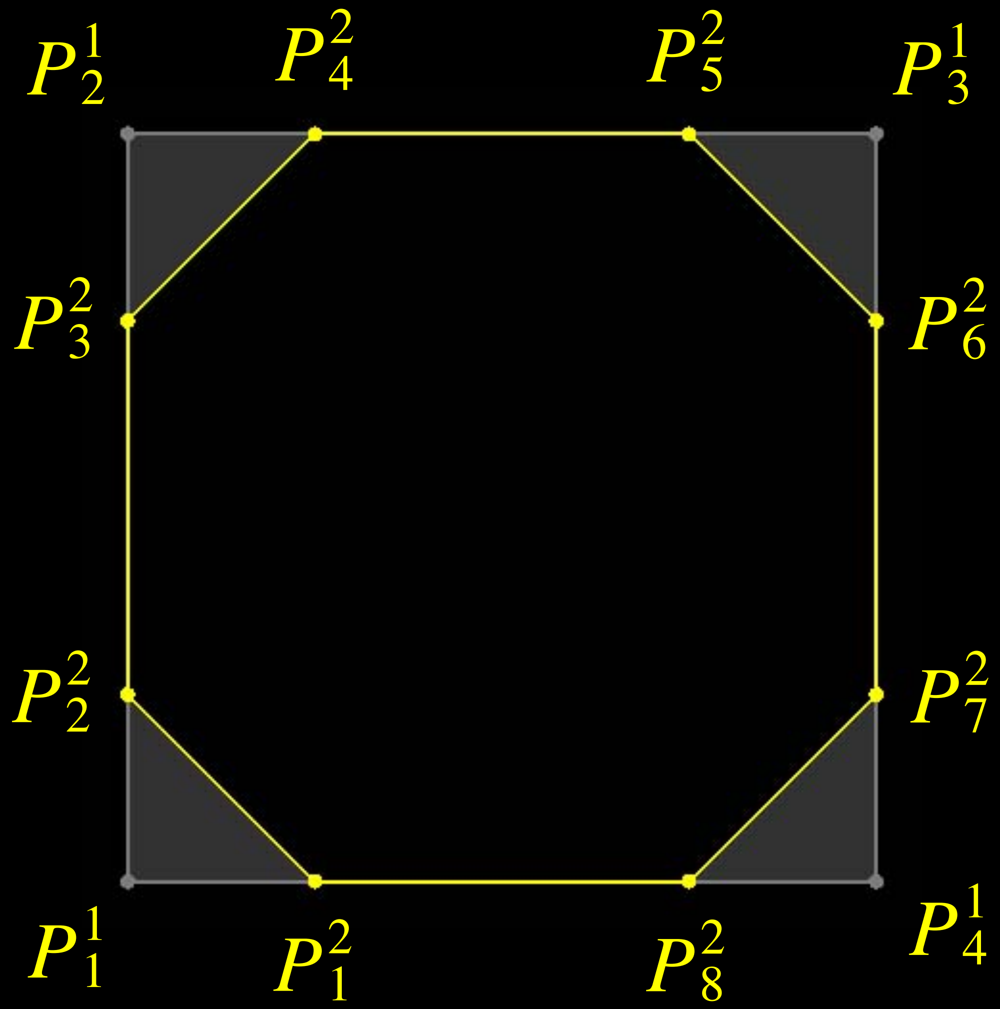
Standard formalization – indexing of points



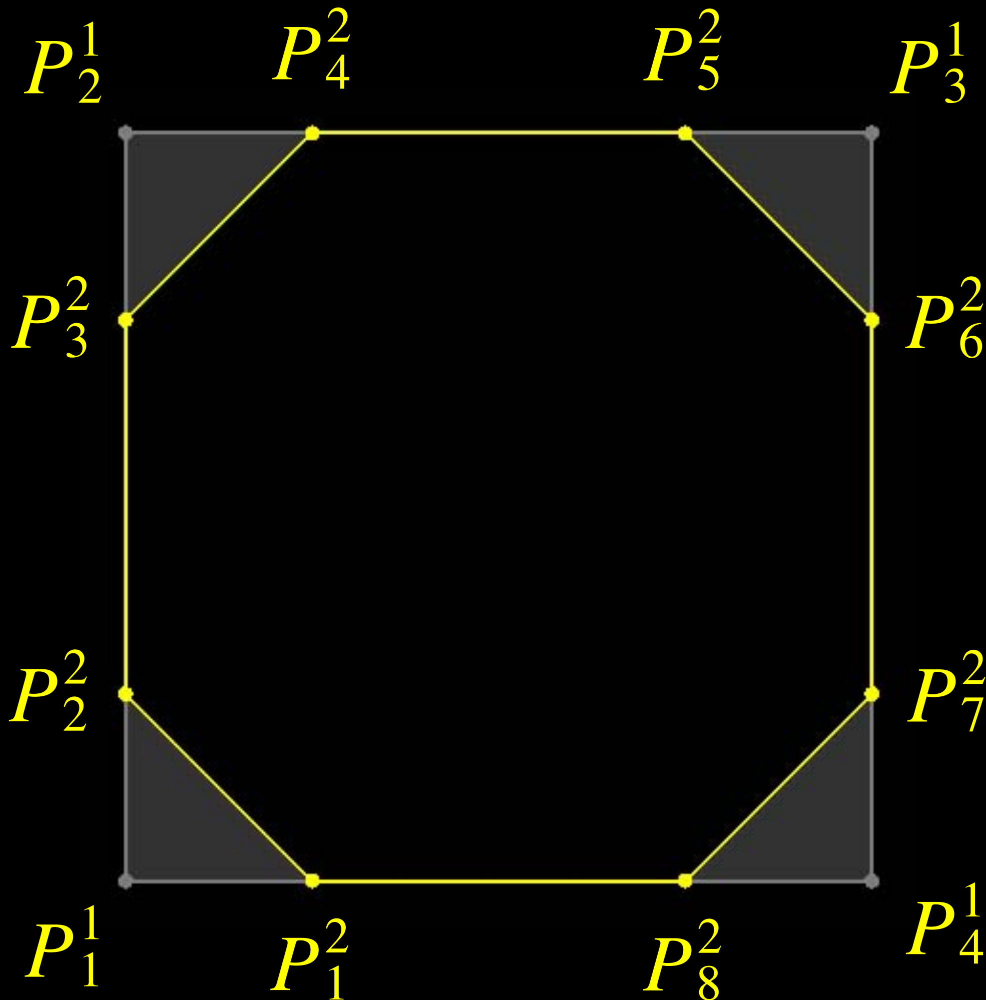
Standard formalization – indexing of points



Standard formalization – indexing of points

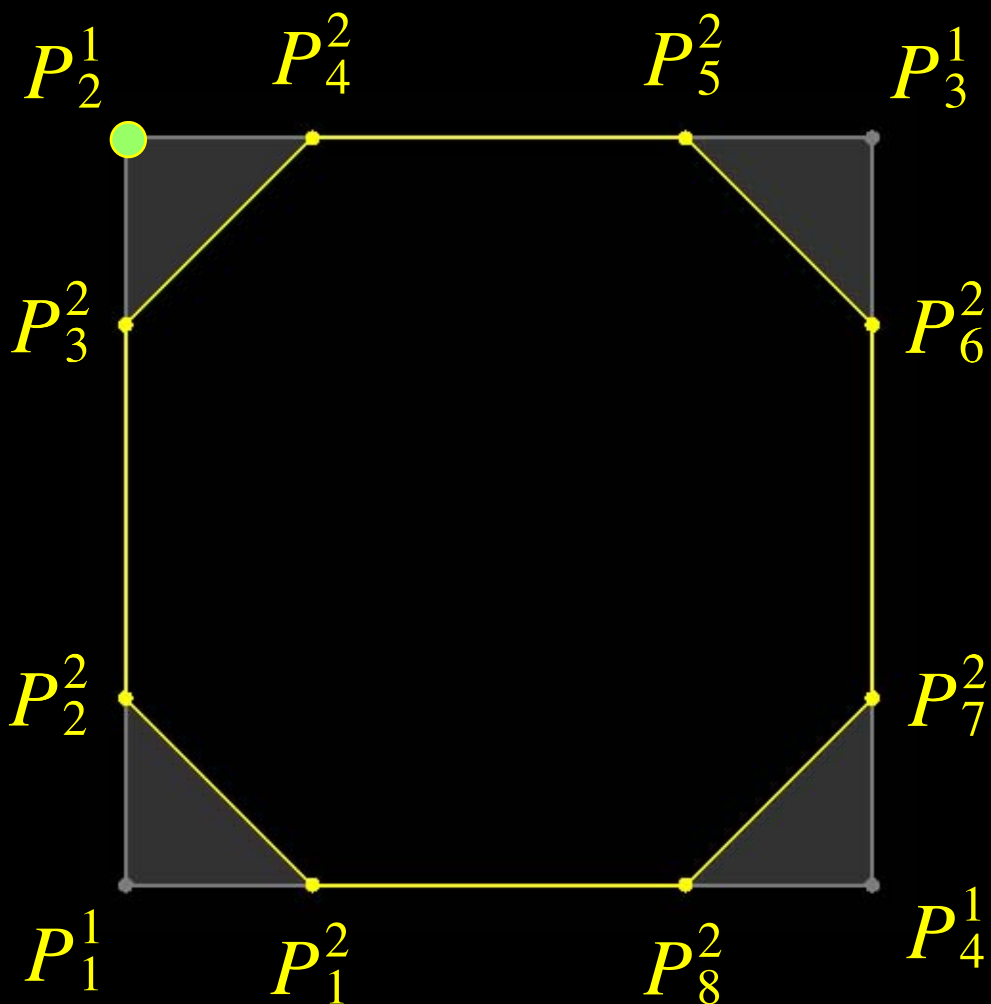


Standard formalization – matrix notation



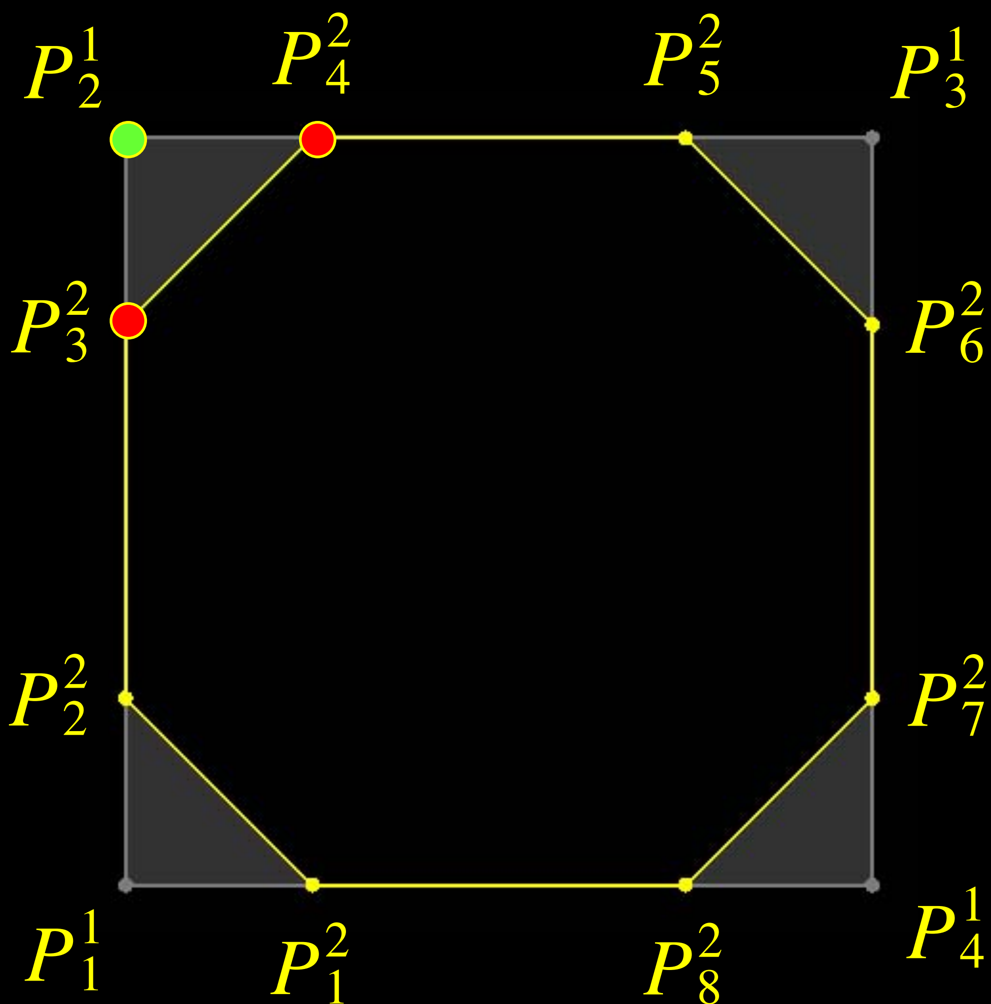
$$\begin{matrix} P_1^2 \\ P_2^2 \\ P_3^2 \\ P_4^2 \\ P_5^2 \\ P_6^2 \\ P_7^2 \\ P_8^2 \end{matrix} = \begin{matrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{matrix} \quad \begin{matrix} P_1^1 \\ P_2^1 \\ P_3^1 \\ P_4^1 \end{matrix}$$

Standard formalization – matrix notation



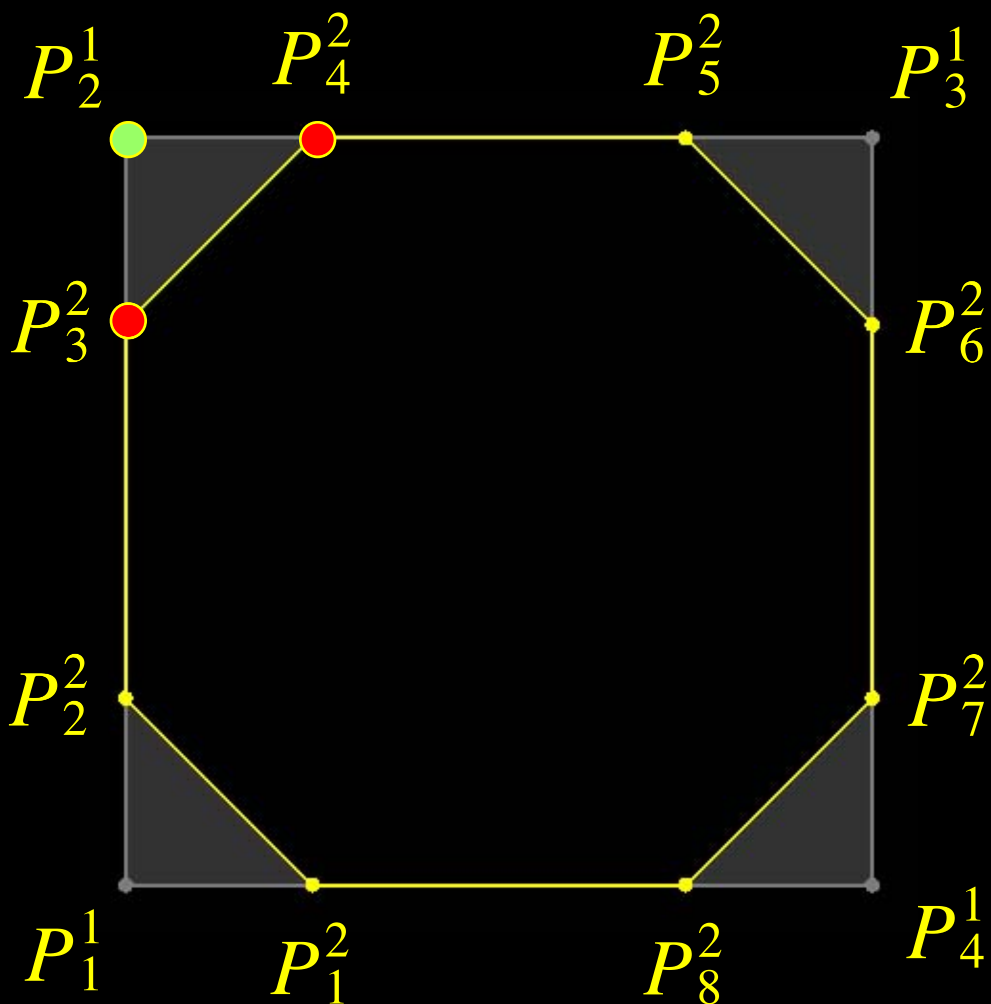
$$\begin{bmatrix} P_1^2 \\ P_2^2 \\ P_3^2 \\ P_4^2 \\ P_5^2 \\ P_6^2 \\ P_7^2 \\ P_8^2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} P_1^1 \\ P_2^1 \\ P_3^1 \\ P_4^1 \end{bmatrix}$$

Standard formalization – matrix notation



$$
\begin{pmatrix} P_1^2 \\ P_2^2 \\ P_3^2 \\ P_4^2 \\ P_5^2 \\ P_6^2 \\ P_7^2 \\ P_8^2 \end{pmatrix}
=
\begin{pmatrix}
\frac{3}{4} & 0 & 0 & \frac{1}{4} \\
\frac{3}{4} & \frac{1}{4} & 0 & 0 \\
\frac{1}{4} & \frac{3}{4} & 0 & 0 \\
0 & \frac{3}{4} & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & \frac{3}{4} & 0 \\
0 & 0 & \frac{3}{4} & \frac{1}{4} \\
0 & 0 & \frac{1}{4} & \frac{3}{4} \\
\frac{1}{4} & 0 & 0 & \frac{3}{4}
\end{pmatrix}
\begin{pmatrix} P_1^1 \\ P_2^1 \\ P_3^1 \\ P_4^1 \end{pmatrix}
$$

Standard formalization – matrix notation



$$\begin{pmatrix} P_1^2 \\ P_2^2 \\ P_3^2 \\ P_4^2 \\ P_5^2 \\ P_6^2 \\ P_7^2 \\ P_8^2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} P_1^1 \\ P_2^1 \\ P_3^1 \\ P_4^1 \end{pmatrix}$$

Standard formalization – matrix notation

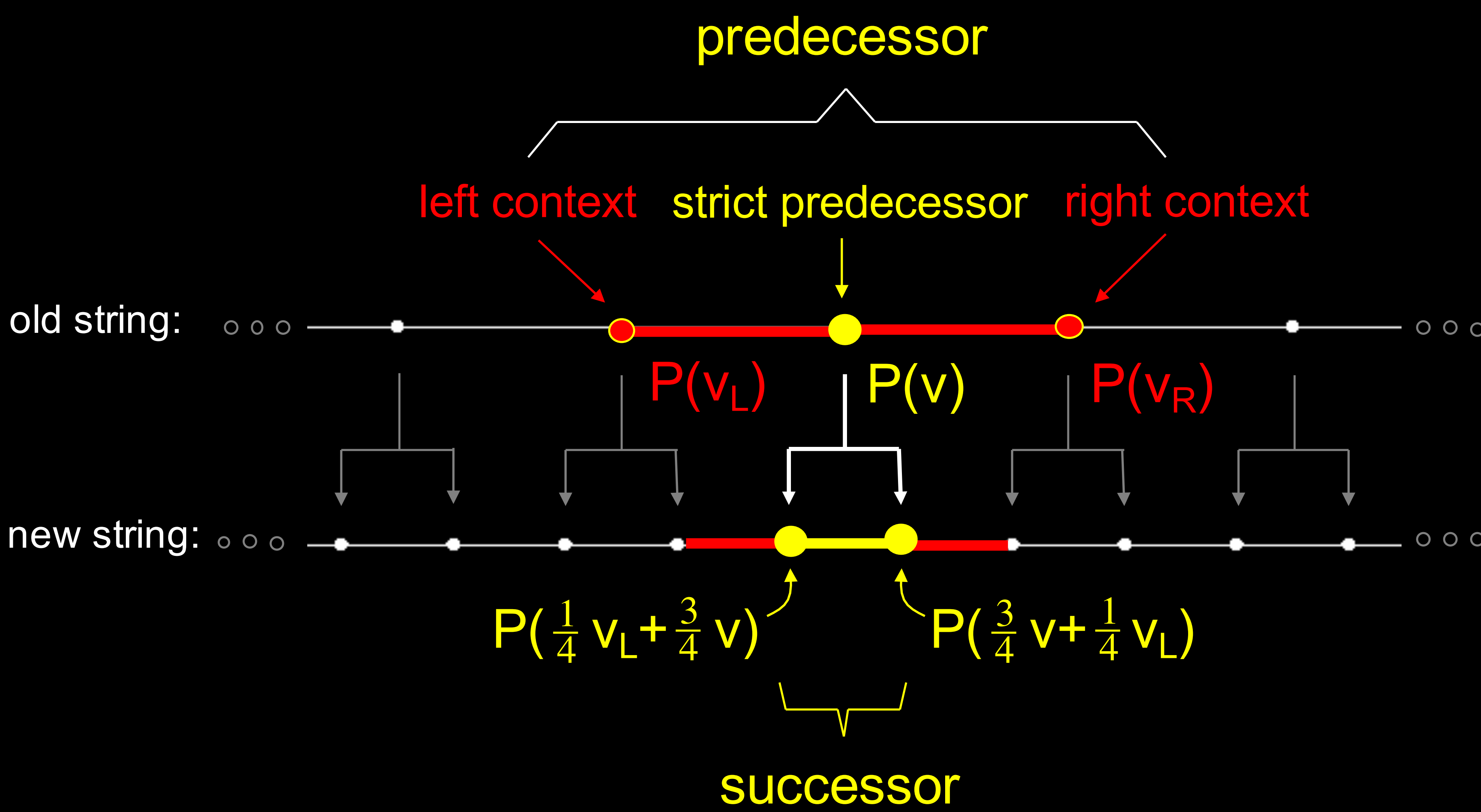
$$
\begin{array}{c}
P_1^{k+1} \\ P_2^{k+1} \\ P_3^{k+1} \\ P_4^{k+1} \\ P_5^{k+1} \\ P_6^{k+1} \\ \vdots \\ P_{2n\ \ 2}^{k+1} \\ P_{2n\ \ 1}^{k+1} \\ P_{2n}^{k+1}
\end{array}
=
\begin{array}{ccccccc}
\frac{3}{4} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{4} \\
\frac{3}{4} & \frac{1}{4} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{4} & \frac{3}{4} & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{3}{4} & \frac{1}{4} & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{4} & \frac{3}{4} & 0 & \cdots & 0 & 0 \\
0 & 0 & \frac{3}{4} & \frac{1}{4} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{3}{4} & \frac{1}{4} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{4} & \frac{3}{4} \\
\frac{1}{4} & 0 & 0 & 0 & \cdots & 0 & \frac{3}{4}
\end{array}
\begin{array}{c}
P_1^k \\ P_2^k \\ P_3^k \\ P_4^k \\ \vdots \\ P_{n\ \ 1}^k \\ P_n^k
\end{array}
$$

Problem

We have a growing matrix:

- not very formal,
- not directly conducive to implementation.

Productions



Formal notation

$$\overbrace{P(v_L)}^{\text{left context}} < \overbrace{P(v)}^{\text{strict prede-cessor}} > \overbrace{P(v_R)}^{\text{right context}} \rightarrow \overbrace{P(\tfrac{1}{4}\, v_L + \tfrac{3}{4}\, v)\; P(\tfrac{3}{4}\, v + \tfrac{1}{4}\, v_R)}^{\text{successor}}$$

left context separator

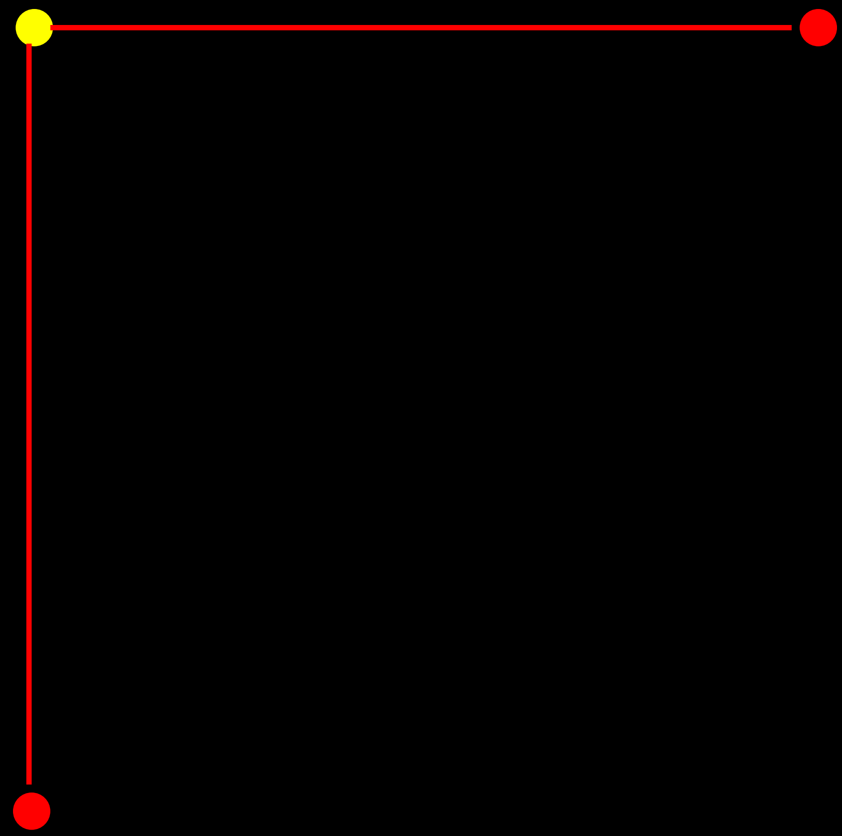
right context separator

production symbol

affine geometry operations

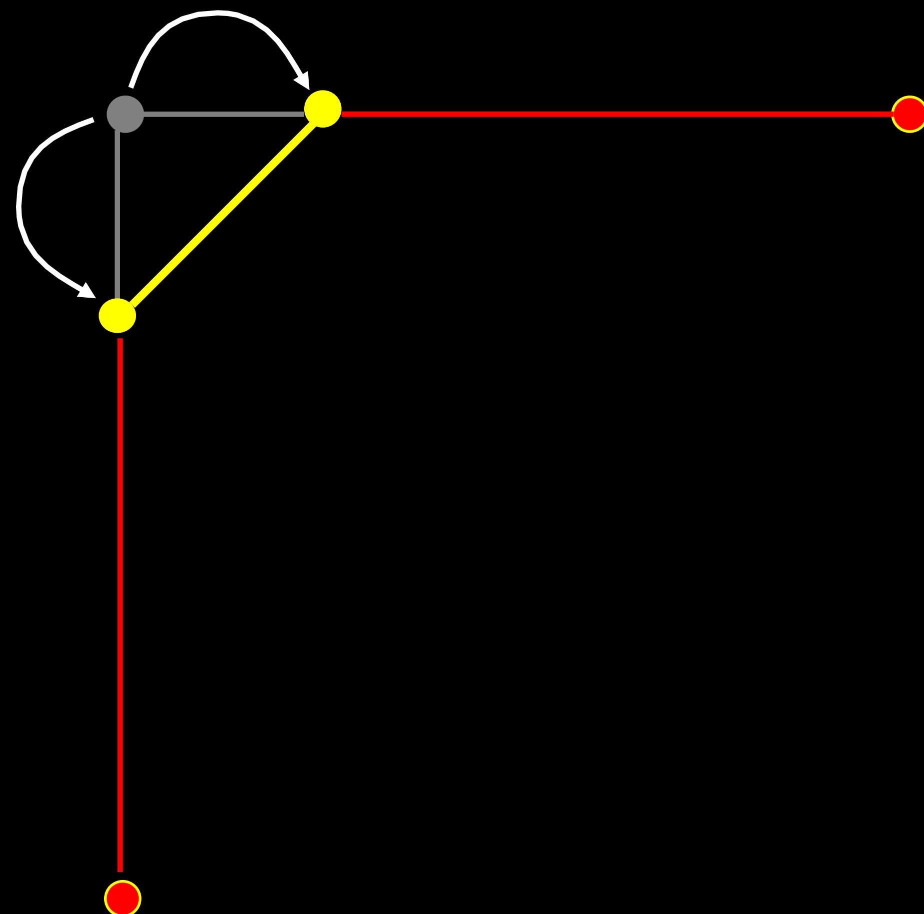
This is a context-sensitive L-system production with the affine geometry interpretation.

Chaikin subdivision in L-system notation



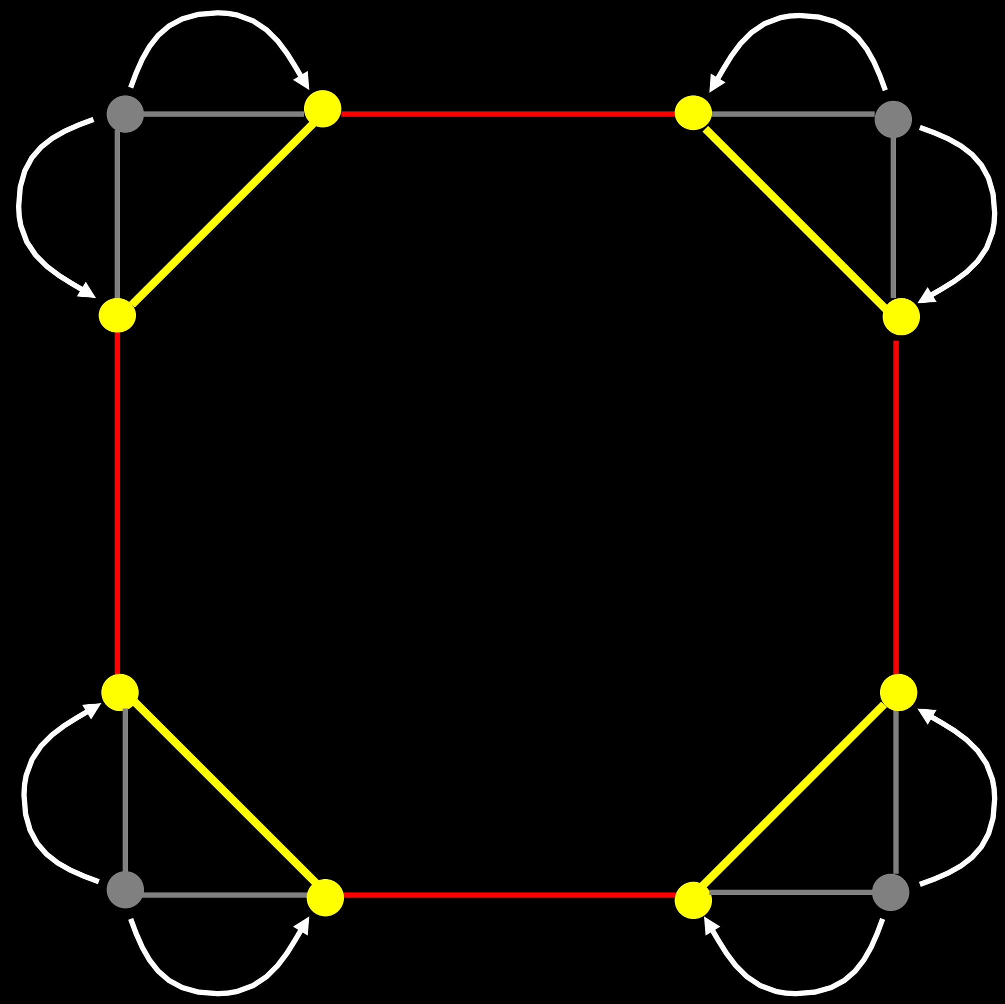
$$\mathrm{P}(v_L)\ \mathrm{E} < \mathrm{P}(v) > \mathrm{E}\ \mathrm{P}(v_R)$$

Chaikin subdivision in L-system notation



$$P(v_L)\ E < P(v) > E\ P(v_R) \rightarrow P\left(\frac{1}{4}v_L + \frac{3}{4}v\right) E\ P\left(\frac{3}{4}v + \frac{1}{4}v_R\right)$$

Chaikin subdivision in L-system notation

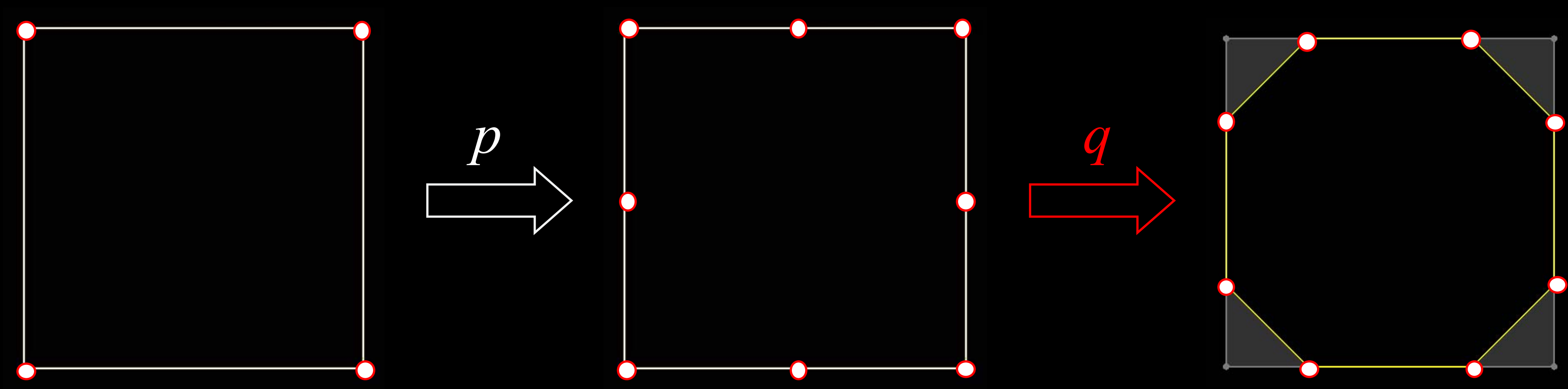


$$\mathrm{P}(v_L)\ \mathrm{E} < \mathrm{P}(v) > \mathrm{E}\ \mathrm{P}(v_R) \rightarrow \mathrm{P}\left(\frac{1}{4}v_L + \frac{3}{4}v\right)\ \mathrm{E}\ \mathrm{P}\left(\frac{3}{4}v + \frac{1}{4}v_R\right)$$

Chaikin subdivision in L+C

```
1.  #include <lpfgall.h>
2.  V2f v1(0, 0), v2(0, 1), v3(1, 1), v4(1, 0);
3.  module P(V2f), E;
4.  ring L-system: 1;
5.  derivation length: 6;
6.  Axiom: P(v1) E P(v2) E P(v3) E P(v4) E ;
7.  P(vl) E  < P(v) >  E P(vr) :
            { produce P(0.25*vl+0.75*v)  E P(0.75*v+0.25*vr);  }
8.  interpretation:
9.  P(v) :  { produce MoveTo2f(v) Circle(0.01) ; }
10  P(vl) < E > P(vr) : { produce Line2f(v) ; }
```

Chaikin algorithm – alternative view



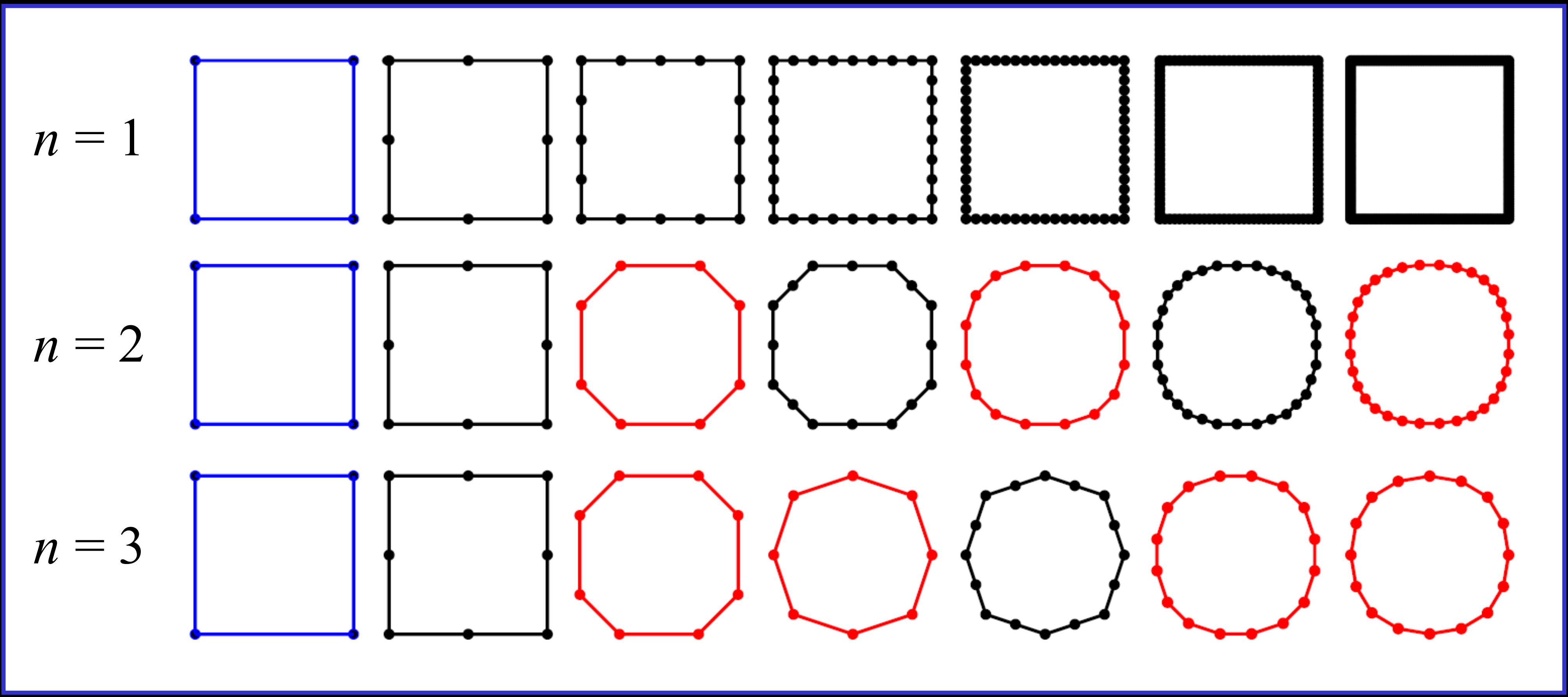
$$p: \quad P(v_L) < E > P(v_R) \rightarrow EP(\tfrac{1}{2}v_L + \tfrac{1}{2}v_R)E$$

$$q_1: \quad P(v_L) < E > P(v_R) \rightarrow P(\tfrac{1}{2}v_L + \tfrac{1}{2}v_R)$$

$$q_2: \quad P(v) \rightarrow E$$

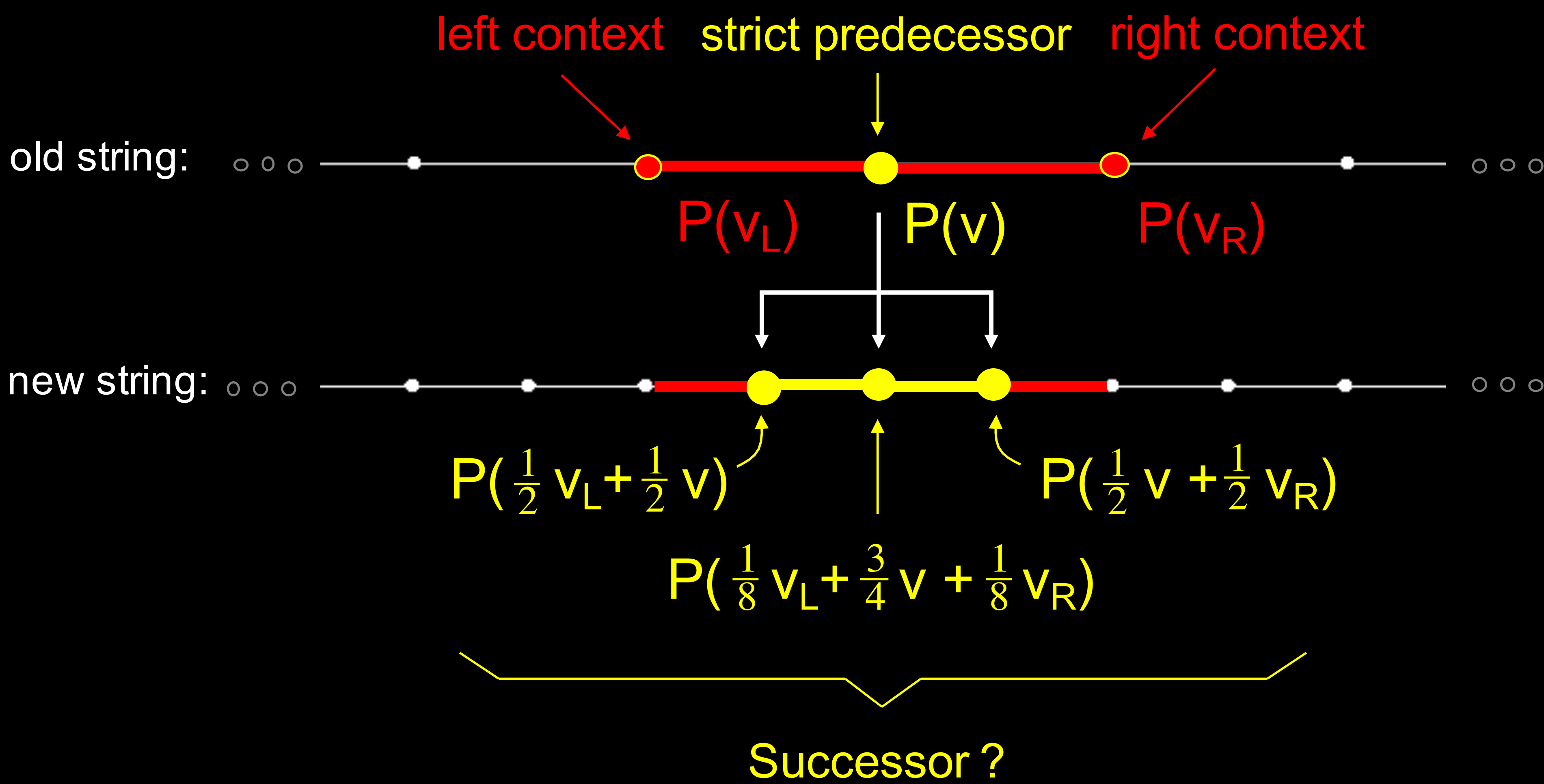
Control language: $L = (pq)^*$

Uniform B-splines: the Lane-Riesenfeld algorithm

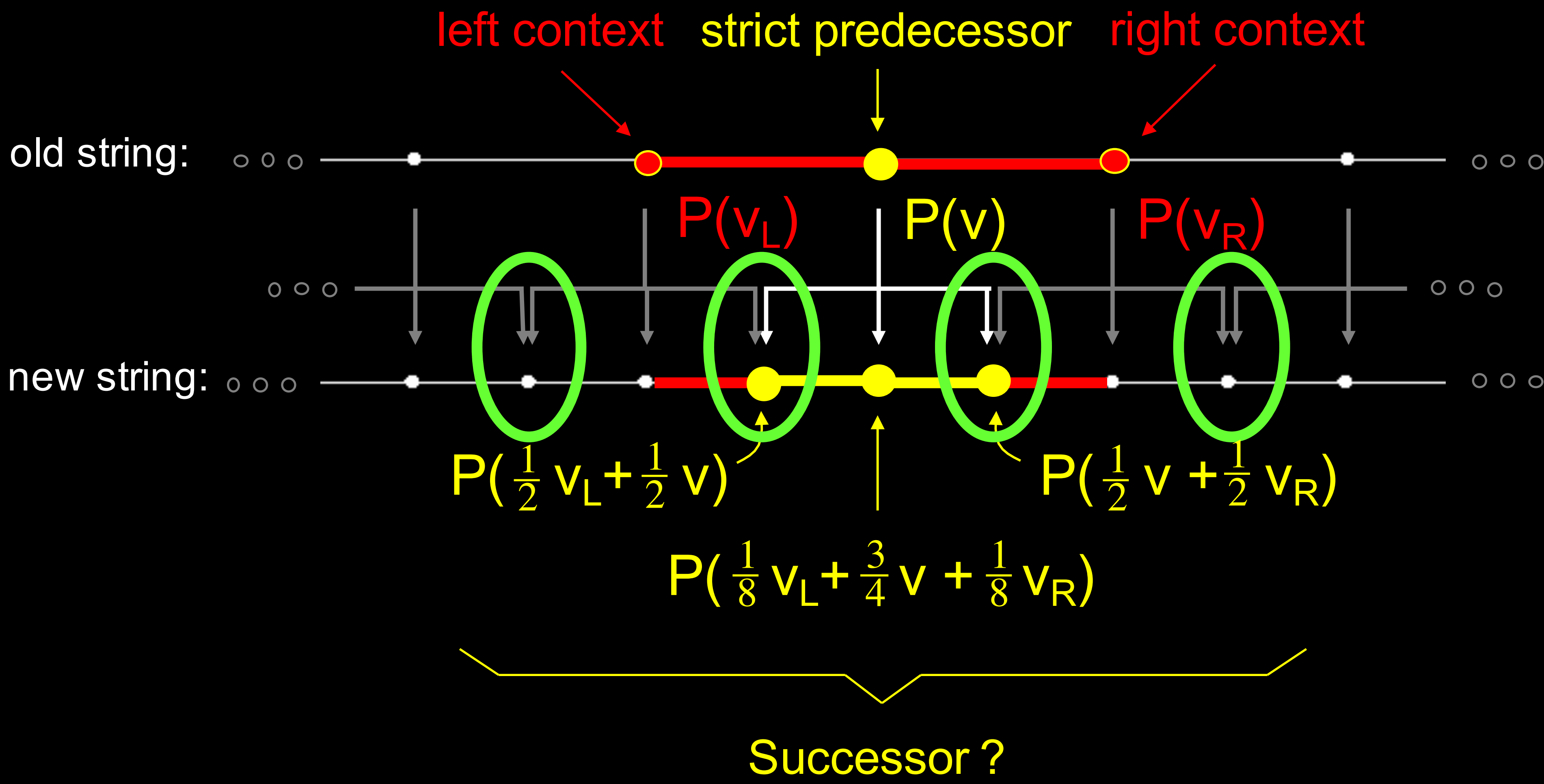


Control language: $L = \left(pq^{n-1}\right)^*$

Cubic B-spline

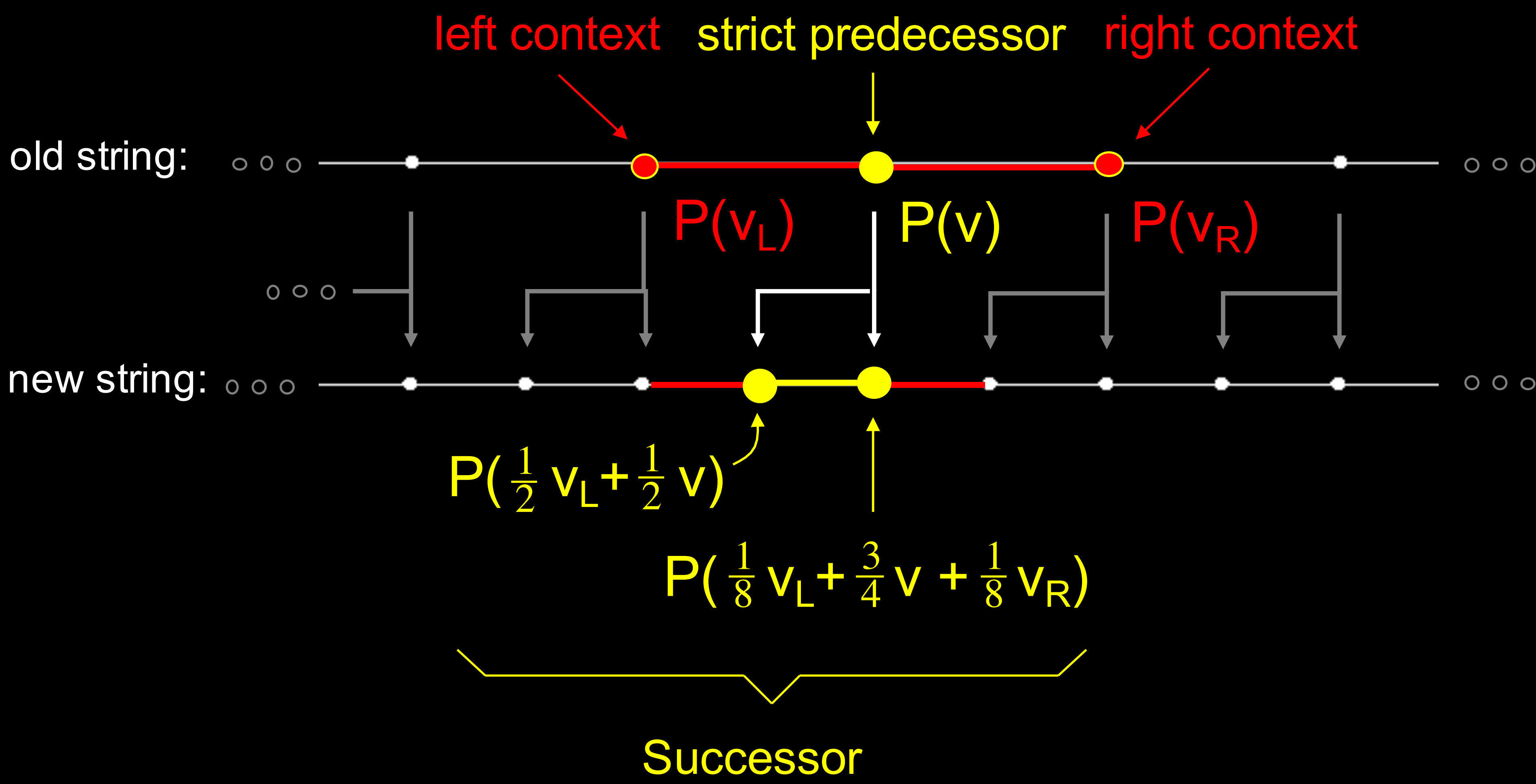


Caveat



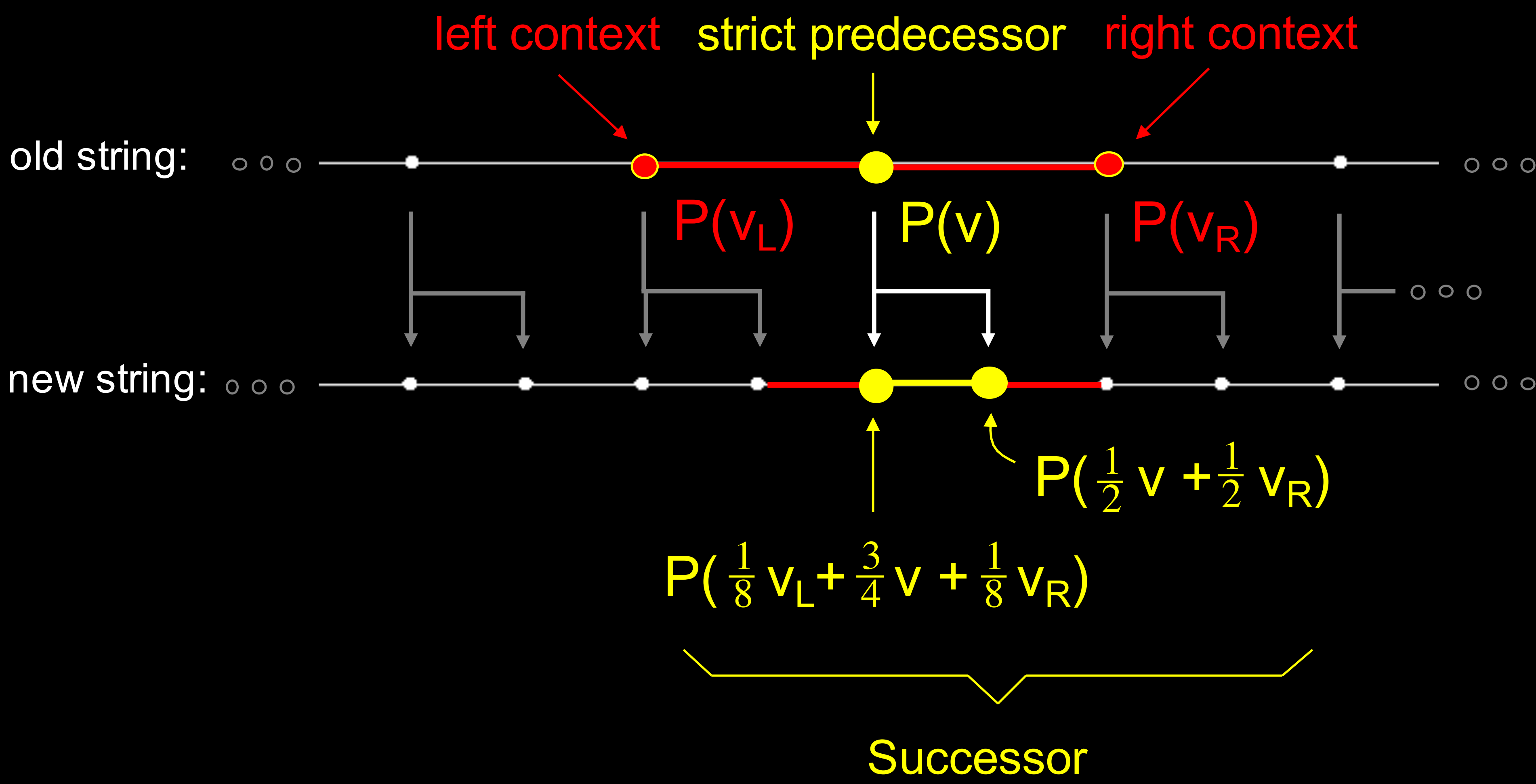
left context
strict predecessor
right context
old string:
new string:
$P(v_L)$
$P(v)$
$P(v_R)$
$P(\frac{1}{2}v_L + \frac{1}{2}v)$
$P(\frac{1}{2}v + \frac{1}{2}v_R)$
$P(\frac{1}{8}v_L + \frac{3}{4}v + \frac{1}{8}v_R)$
Successor ?

Solution 1



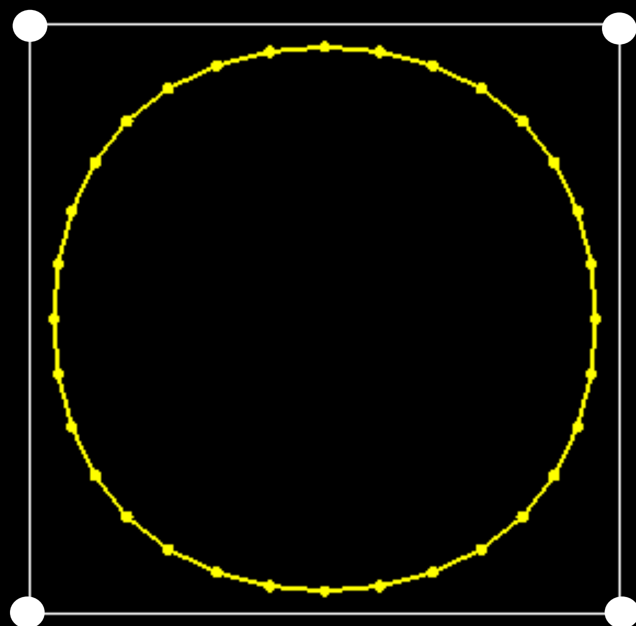
$$P(v_L) < P(v) > P(v_R) \rightarrow P(\tfrac{1}{2}v_L + \tfrac{1}{2}v)P(\tfrac{1}{8}v_L + \tfrac{3}{4}v + \tfrac{1}{8}v_R)$$

Solution 2



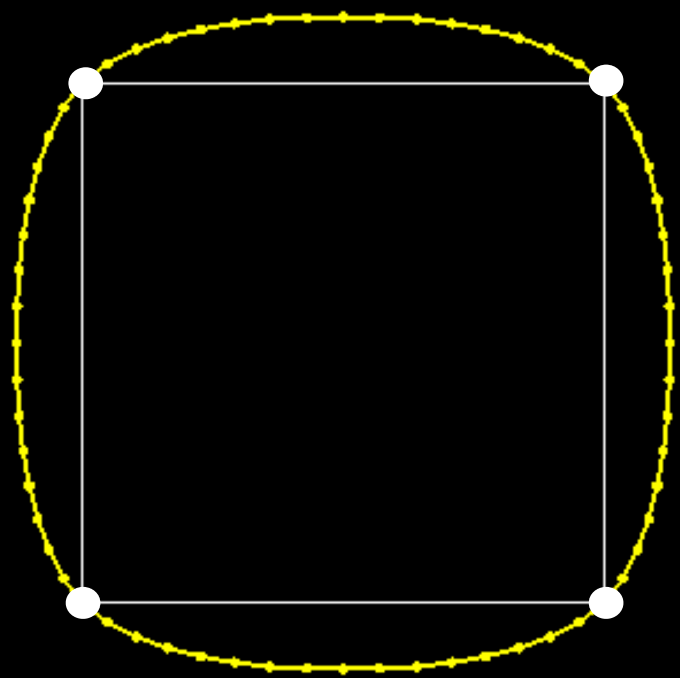
$$P(v_L) < P(v) > P(v_R) \rightarrow P(\tfrac{1}{8}v_L + \tfrac{3}{4}v + \tfrac{1}{8}v_R)P(\tfrac{1}{2}v + \tfrac{1}{2}v_R)$$

B-spline



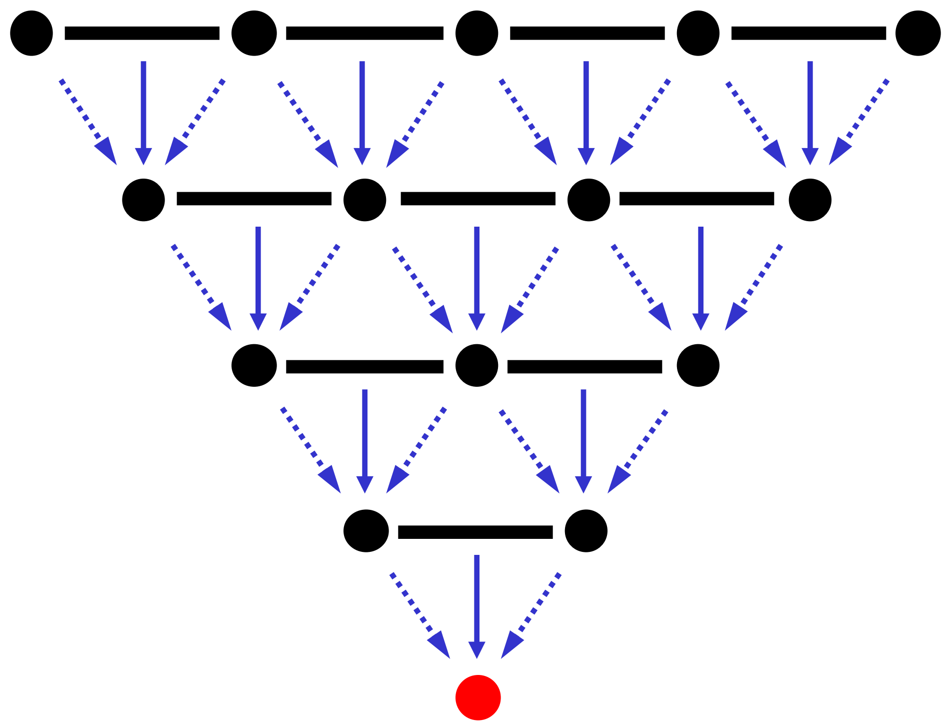
$$P(v_L) < P(v) > P(v_R) \longrightarrow P(\tfrac{1}{8}v_L + \tfrac{3}{4}v + \tfrac{1}{8}v_R)P(\tfrac{1}{2}v + \tfrac{1}{2}v_R)$$

Dyn-Levin-Gregory (4-point subdivision)



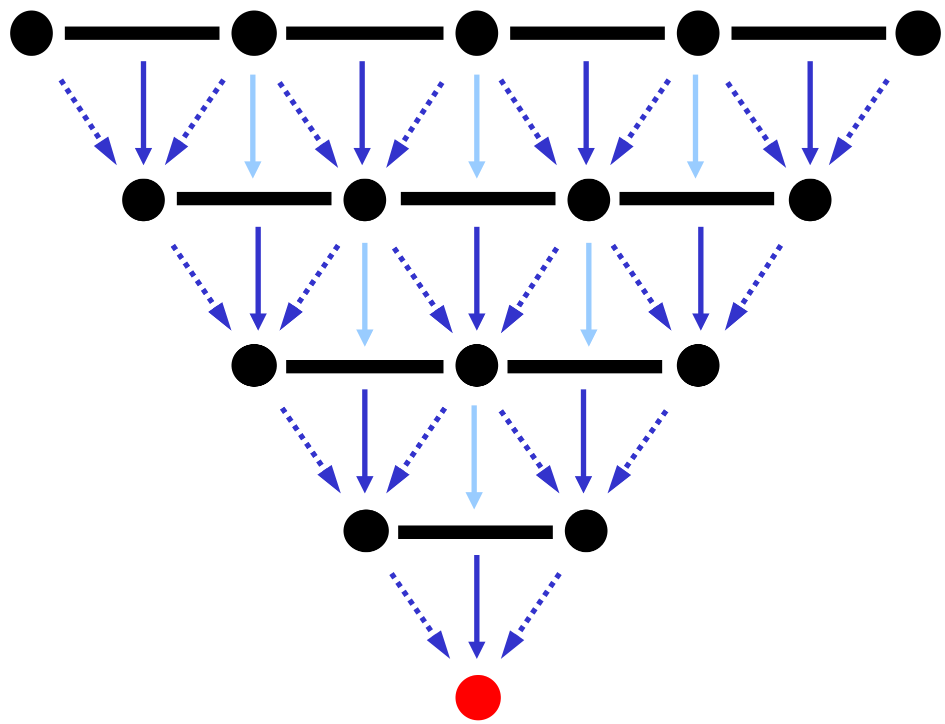
$$P(v_{LL})P(v_L) < P(v) > P(v_R) \rightarrow P(-\tfrac{1}{16}v_{LL} + \tfrac{9}{16}v_L + \tfrac{9}{16}v - \tfrac{1}{16}v_R)P(v)$$

The de Casteljau algorithm



$$P(v_L) < E > P(v_R) \rightarrow P((1\text{-}t)v_L + tv_R)$$

The de Casteljau algorithm



p_1: $P(v_L) < E > P(v_R) \rightarrow P((1\text{-}t)v_L + tv_R)$

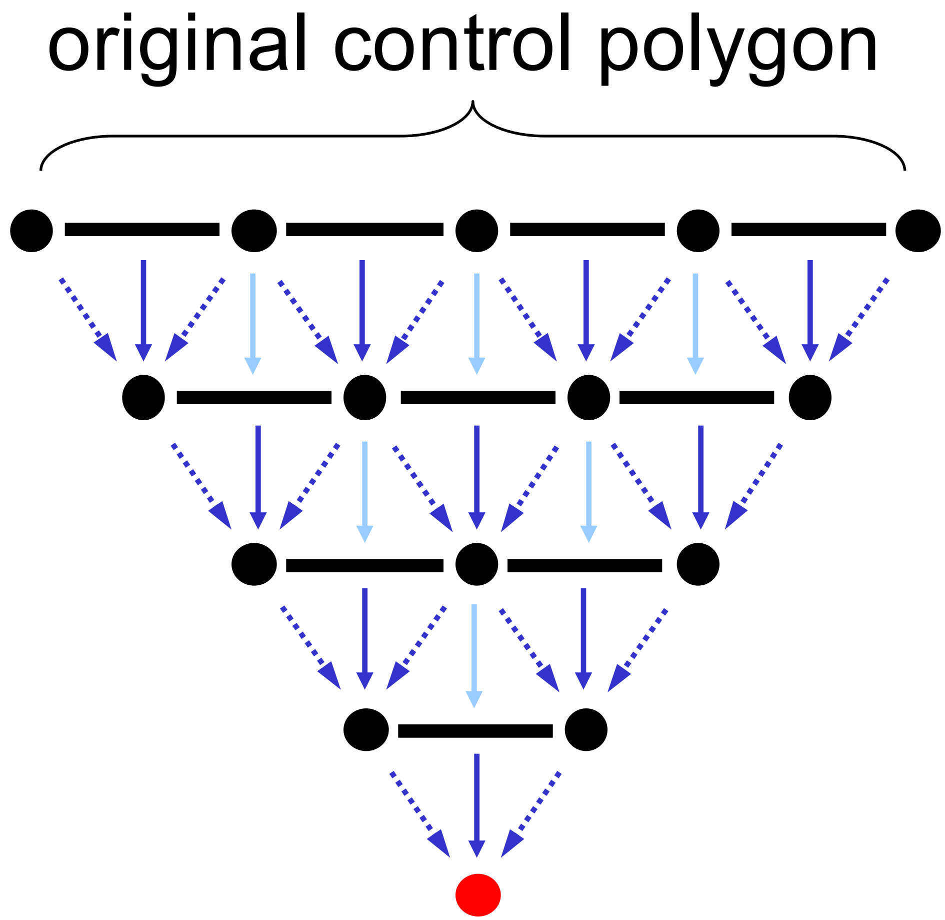
p_2: $E < P(v) > E \rightarrow E$

p_3: $P(v) \rightarrow \varepsilon$

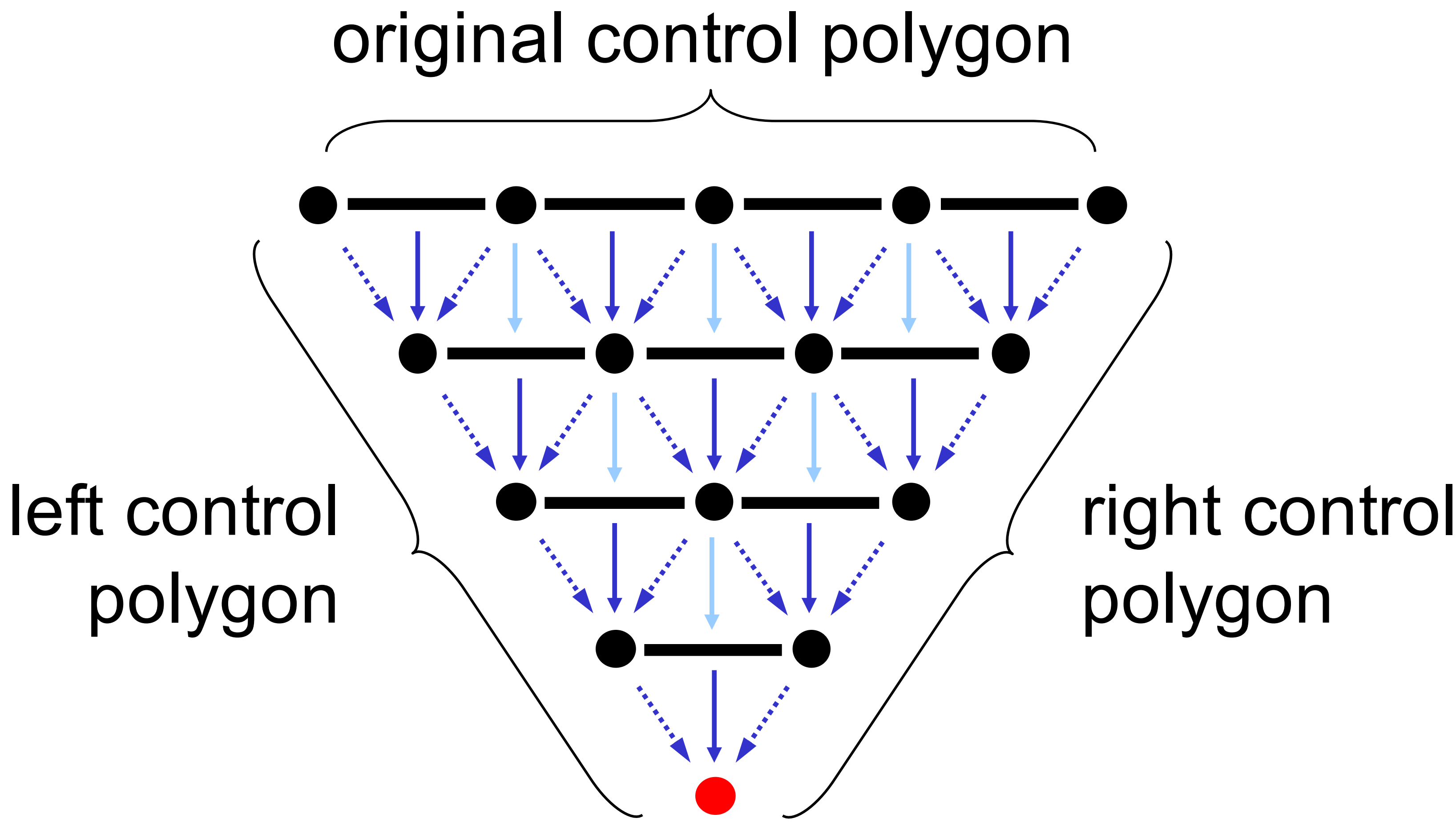
The de Casteljau algorithm

Quadratic	Quadratic
Cubic	Cubic
Quartic	Quartic
Parameter *t* manipulation	Control point manipulation

The de Casteljau subdivision



The de Casteljau subdivision



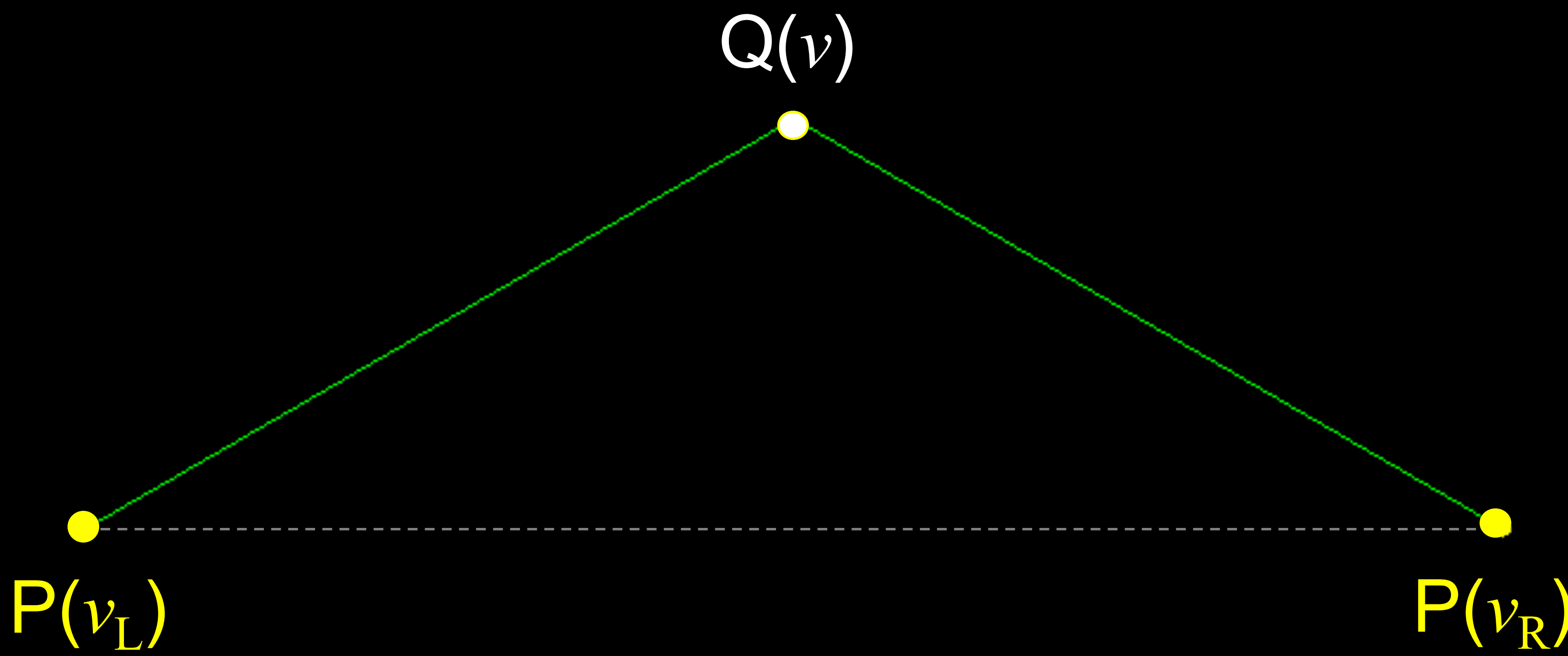
Bezier curve generation with subdivision

Arbitrary

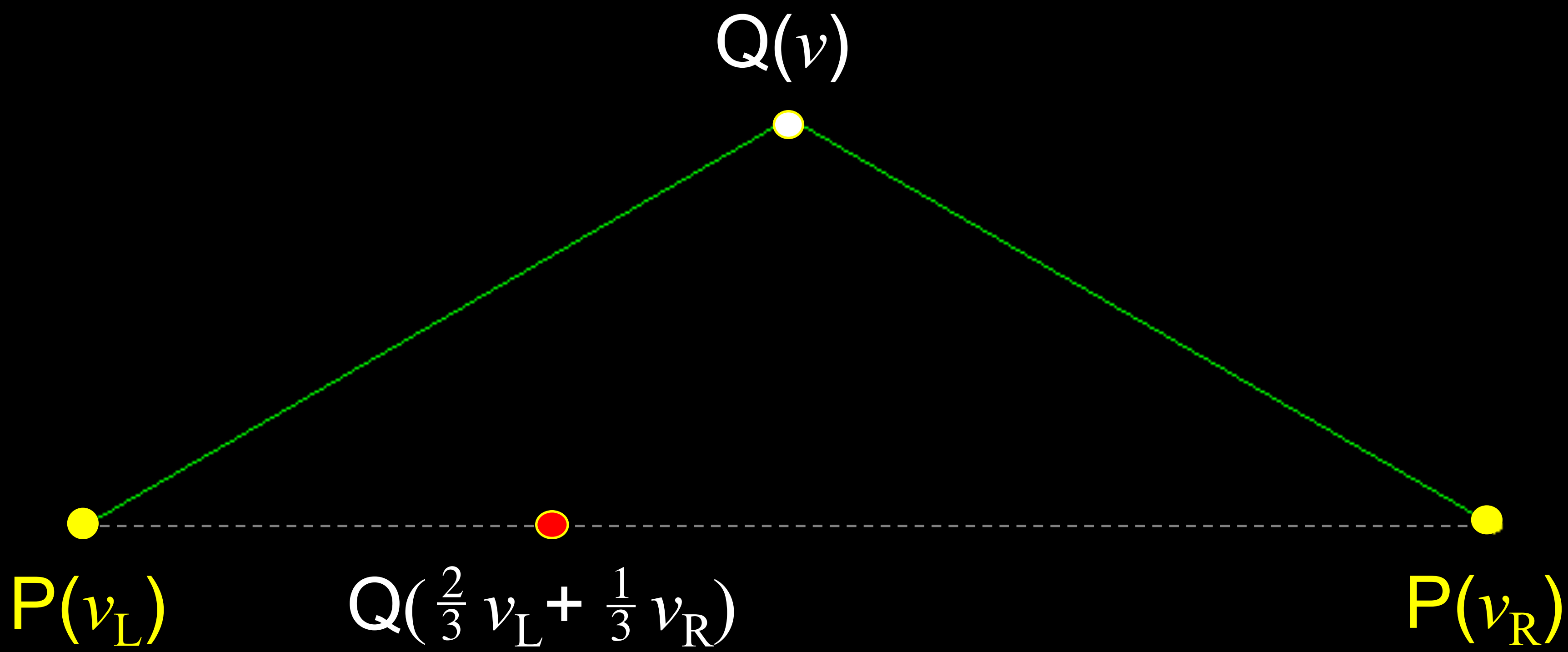
Quadratic

Cubic

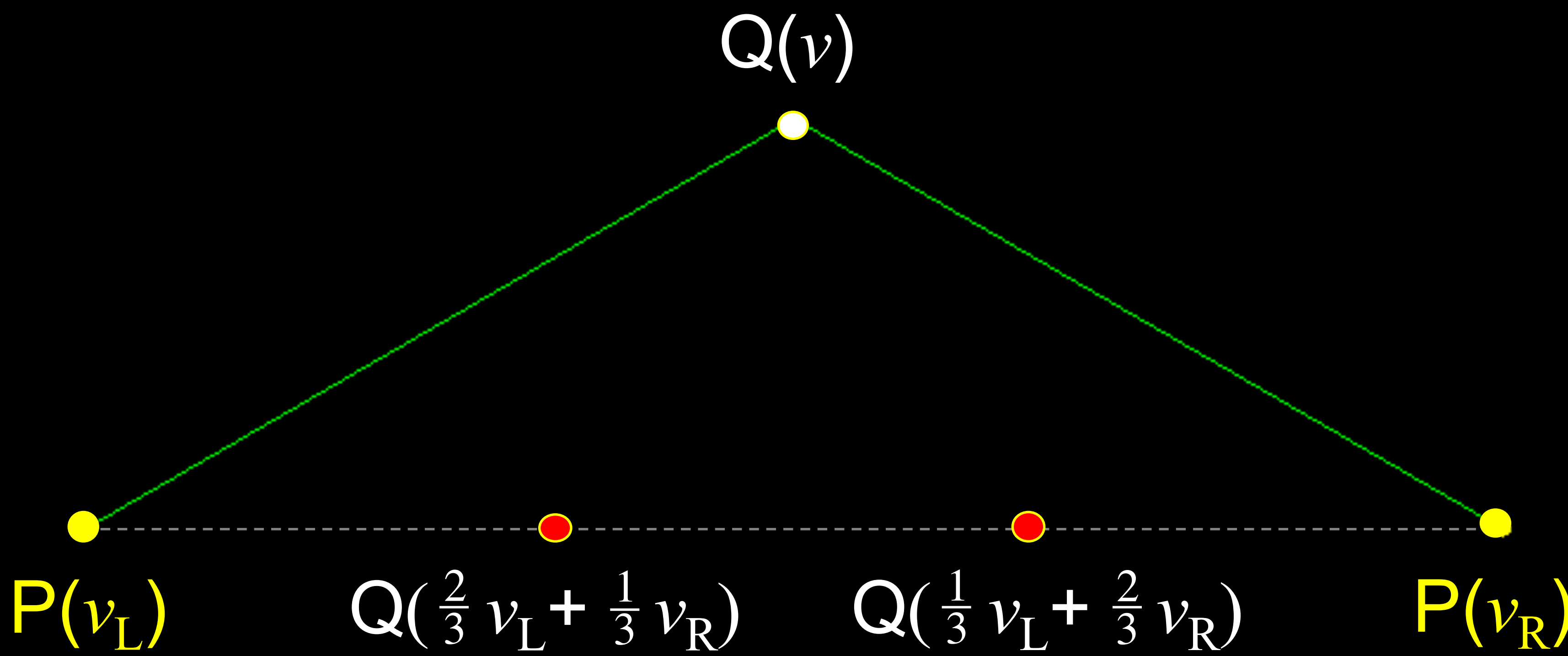
Snowflake curve – Cesaro construction



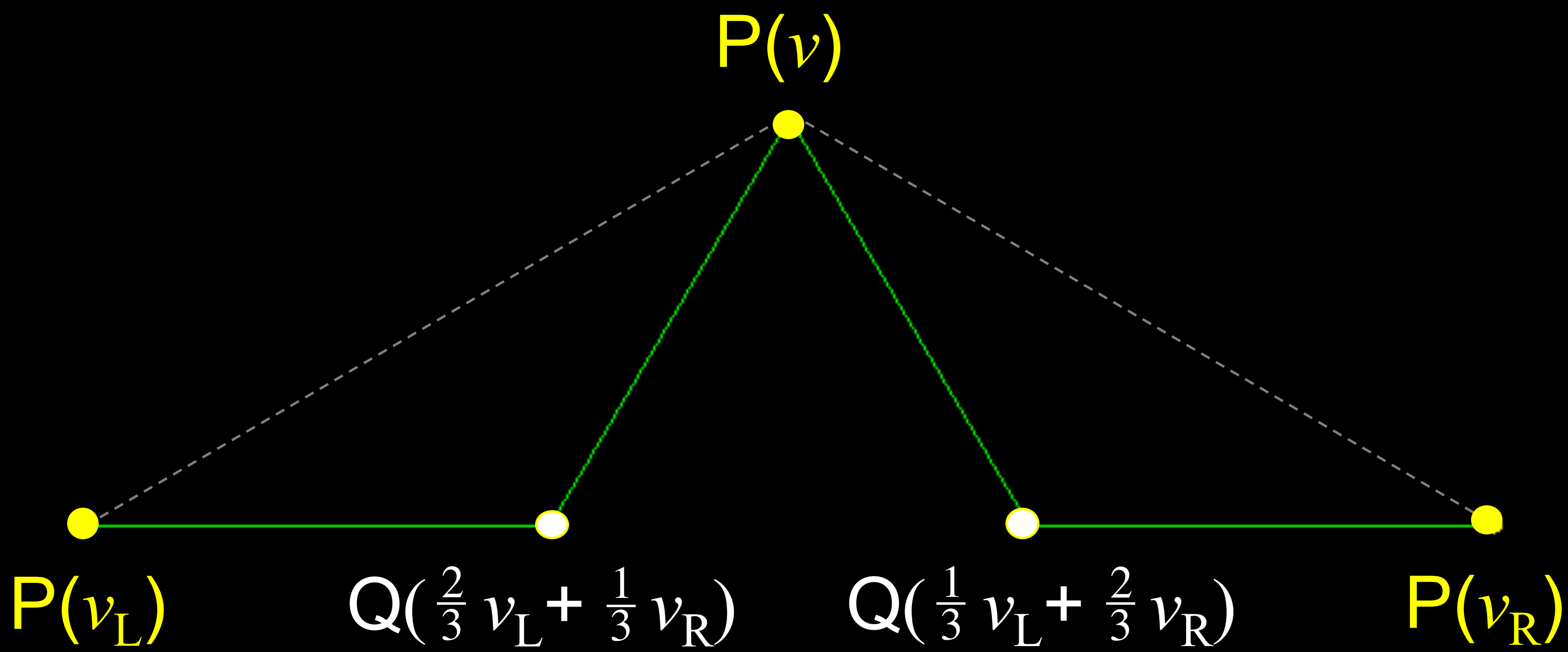
Snowflake curve – Cesaro construction



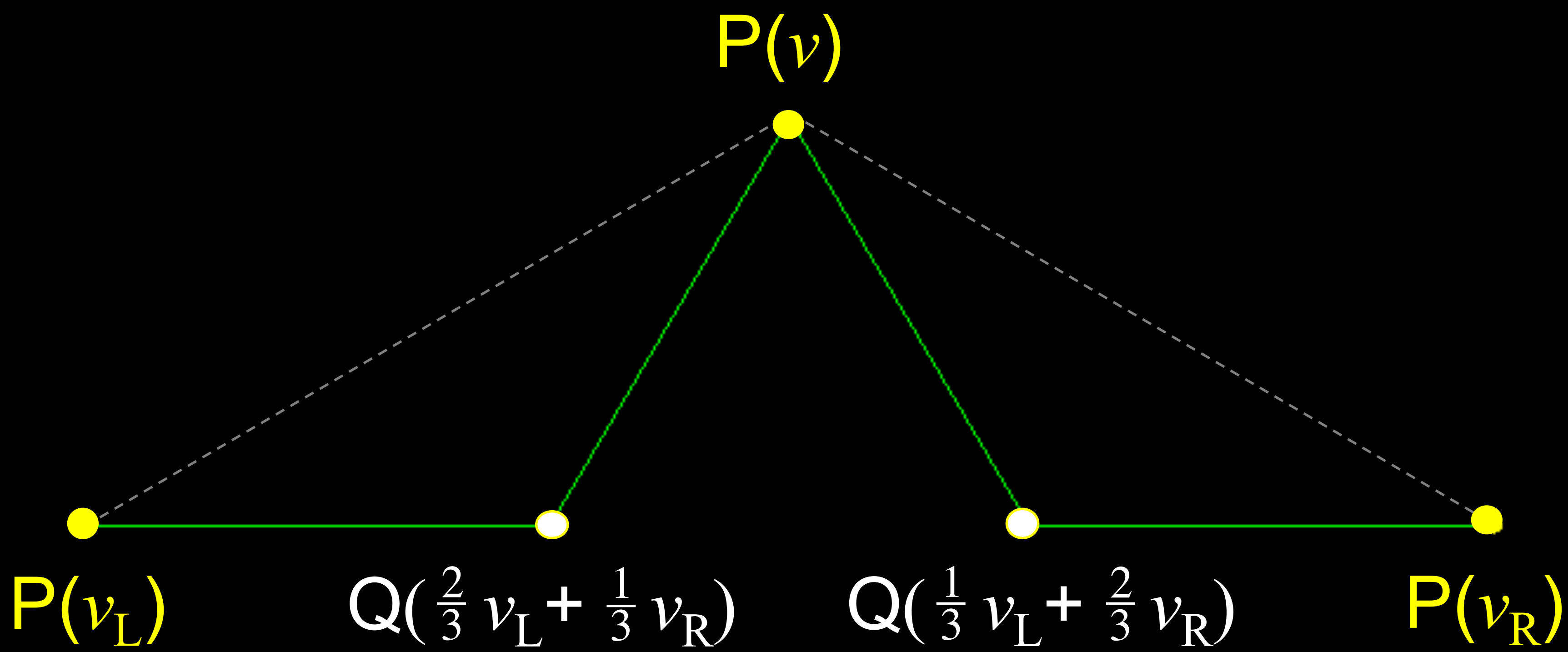
Snowflake curve – Cesaro construction



Snowflake curve – Cesaro construction

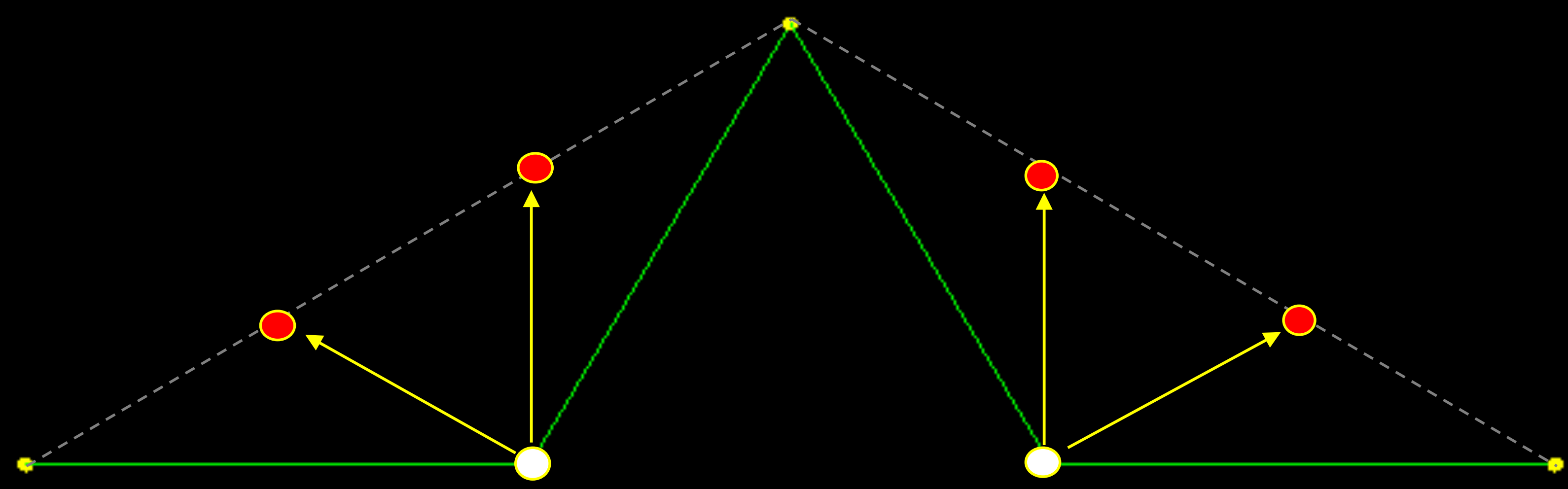


Snowflake curve – Cesaro construction



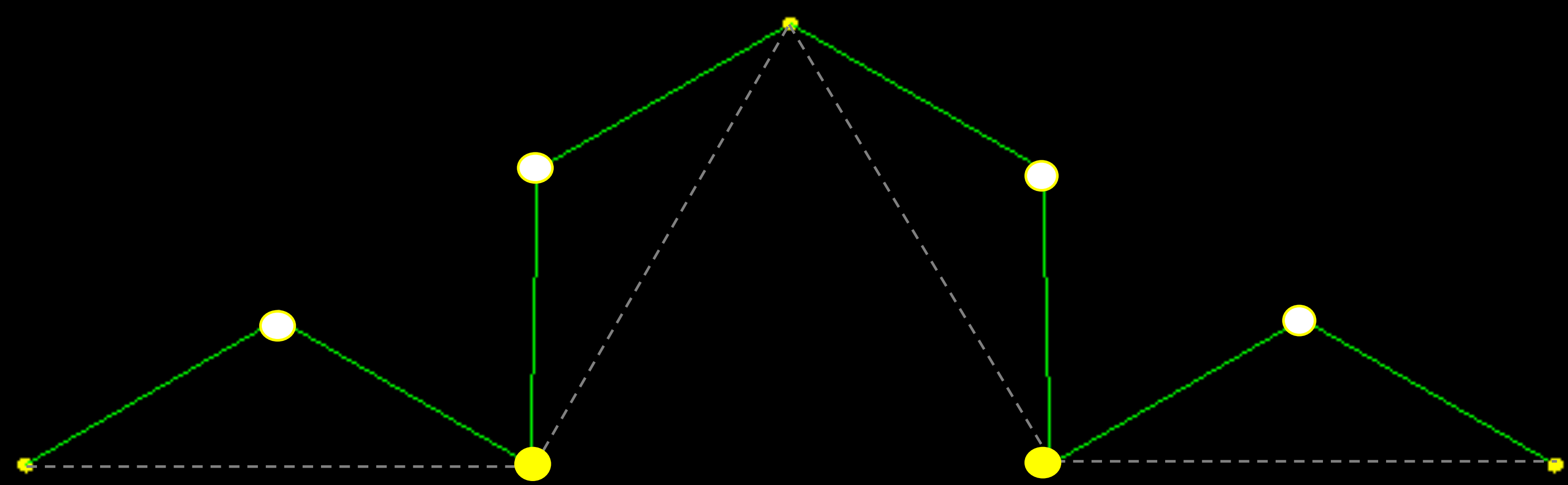
$$P(v_L) < Q(v) > P(v_R) \rightarrow Q(\tfrac{2}{3}v_L + \tfrac{1}{3}v_R)\ P(v)\ Q(\tfrac{2}{3}v_L + \tfrac{1}{3}v_R)$$

Snowflake curve – Cesaro construction



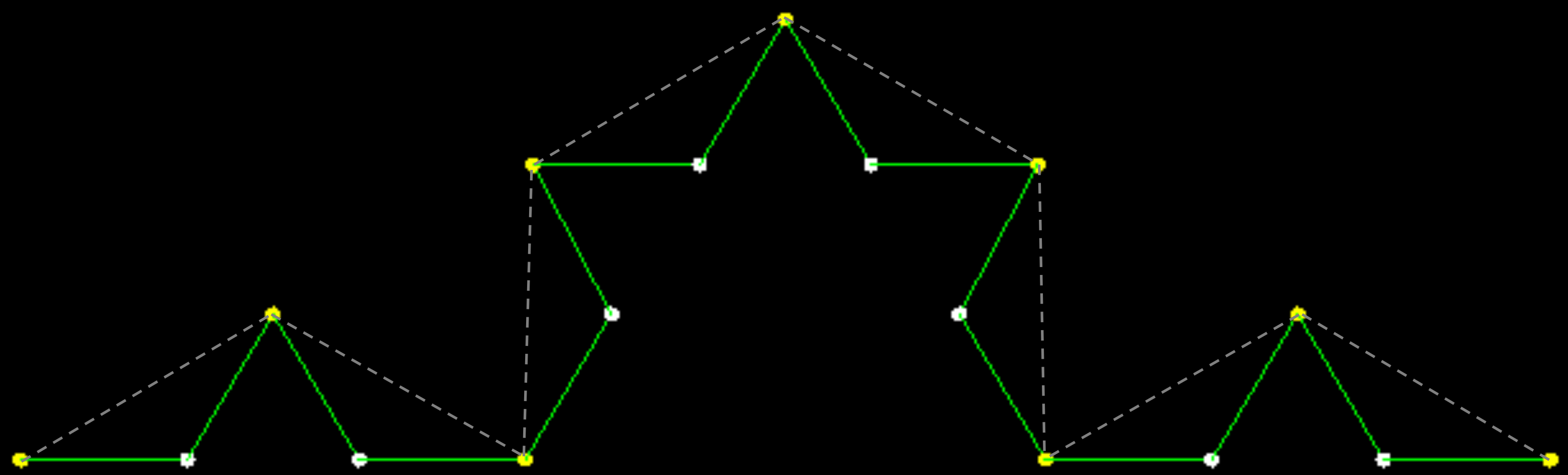
$$P(v_L) < Q(v) > P(v_R) \rightarrow Q\left(\tfrac{2}{3}v_L + \tfrac{1}{3}v_R\right)\ P(v)\ Q\left(\tfrac{2}{3}v_L + \tfrac{1}{3}v_R\right)$$

Snowflake curve – Cesaro construction



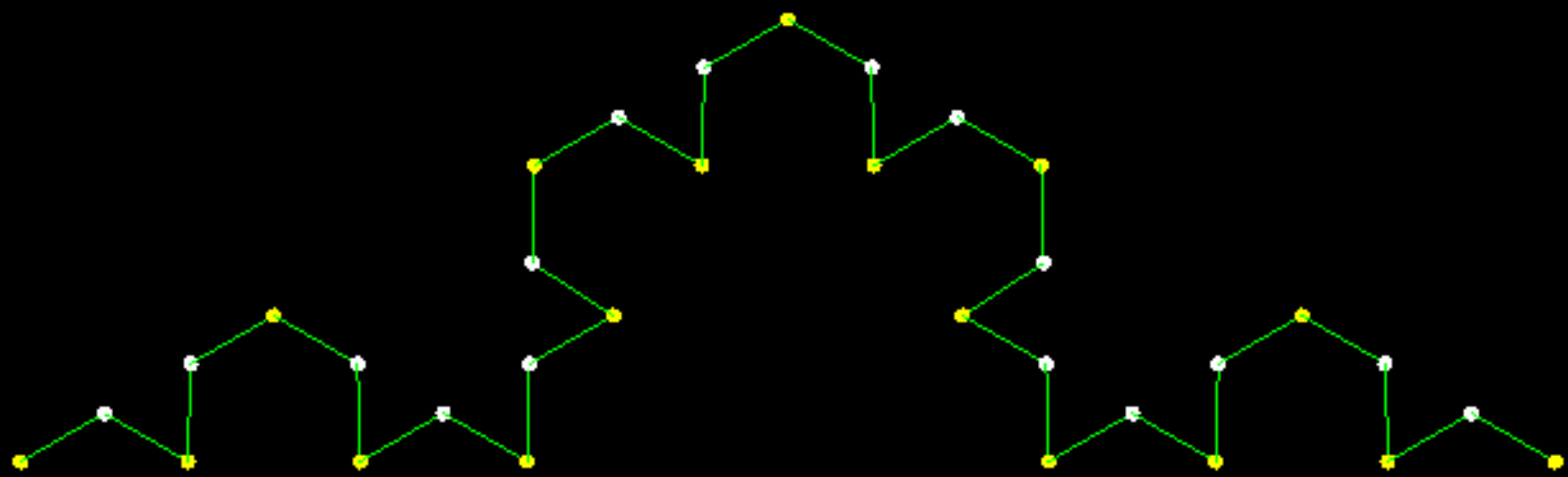
$$P(v_L) < Q(v) > P(v_R) \rightarrow Q(\tfrac{2}{3}v_L + \tfrac{1}{3}v_R)\ P(v)\ Q(\tfrac{2}{3}v_L + \tfrac{1}{3}v_R)$$

Snowflake curve – Cesaro construction



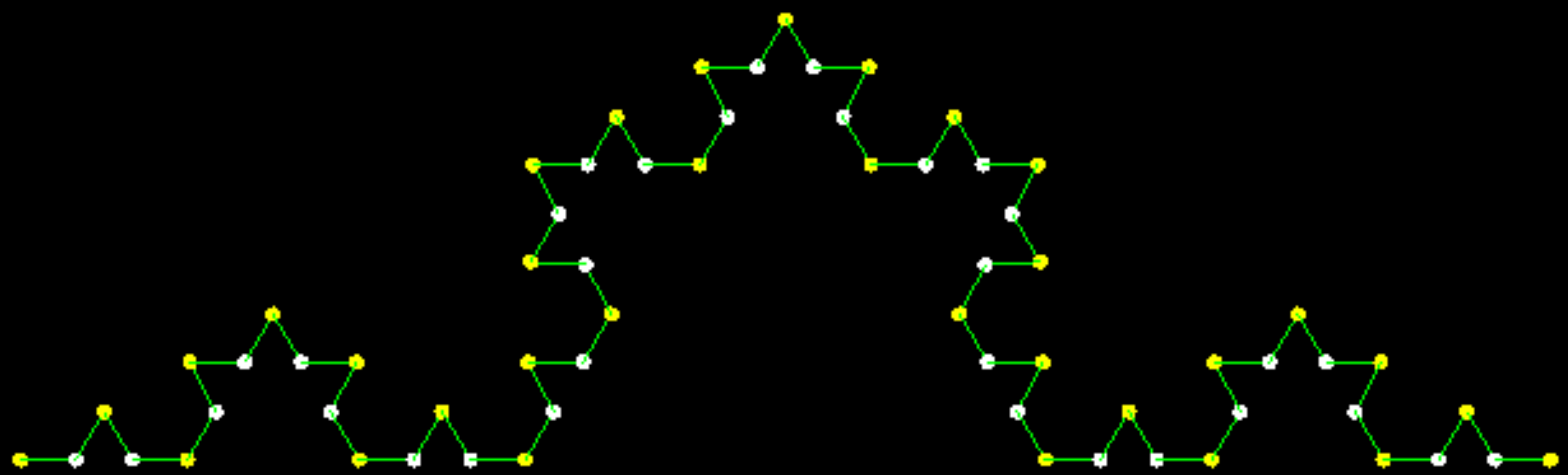
$$P(v_L) < Q(v) > P(v_R) \rightarrow Q(\tfrac{2}{3}v_L + \tfrac{1}{3}v_R)\ P(v)\ Q(\tfrac{2}{3}v_L + \tfrac{1}{3}v_R)$$

Snowflake curve – Cesaro construction



$$P(v_L) < Q(v) > P(v_R) \rightarrow Q(\tfrac{2}{3}v_L + \tfrac{1}{3}v_R)\ P(v)\ Q(\tfrac{2}{3}v_L + \tfrac{1}{3}v_R)$$

Snowflake curve – Cesaro construction



$P(v_L) < Q(v) > P(v_R) \rightarrow Q(\frac{2}{3}v_L + \frac{1}{3}v_R)\ P(v)\ Q(\frac{2}{3}v_L + \frac{1}{3}v_R)$

The End